On the dimension of limit sets on $\mathbb{P}(\mathbb{R}^3)$ via stationary measures

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Contents

§1 Lecture 1: Introduction

Classical results.

Bowen's dimension rigidity. Let $\Gamma = \pi_1(S_g)$ where S_g is a closed surface with genus $g \geqslant 2$. Let

$$\rho_0 : \Gamma \xrightarrow{\eta_0} \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$$

be a representation in $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$. We note that $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}_+(\mathbb{H}^2)$ and $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}_+(\mathbb{H}^3)$. For $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$, we consider the action Γ on $\mathbb{H}^3$ induced by ρ. The limit set of $\rho(\Gamma)$, denoted by $L(\rho(\Gamma))$, is the set of accumulation points of the $\rho(\Gamma)$-orbits, which is on the boundary $\partial\mathbb{H}^3 \cong \mathbb{S}^2$. Here we mention that the action of $\rho(\Gamma)$ on $\mathbb{S}^2$ is conformal.

Example 1.1 $L(\rho_0(\Gamma)) = \partial\mathbb{H}^2$.

Theorem 1.2 (Bowen)
Let ρ be a small perturbation of ρ_0. Then $\dim_{\mathrm{H}}(L(\rho(\Gamma))) \geqslant 1$ and $\dim_{\mathrm{H}}(L(\rho(\Gamma))) = 1$ if and only if there exists g such that $g\rho(\Gamma)g^{-1} \subset \mathrm{PSL}_2(\mathbb{R})$.

Patterson-Sullivan dimension formula. For $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$, let $\delta(\rho)$ be the critical exponent of the Poincaré series

$$P_\rho(s) = \sum_{\gamma\in\Gamma} \left(\frac{\sigma_2(\rho(\gamma))}{\sigma_1(\rho(\gamma))}\right)^s,$$

where $\sigma_1 \geqslant \sigma_2$ are singular values. That is, $\delta(\rho) = \sup_{P_\rho(s)=\infty} s = \inf_{P_\rho(s)<\infty} s$.

Theorem 1.3

Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{C}))$ be a convex cocompact representation (i.e. the action of $\rho(\Gamma)$ on $\mathbb{H}^3$ admits a finite-sided fundamental domain, e.g. $\rho_0 : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$). Then we have

$$\dim_{\mathrm{H}} L(\rho(\Gamma)) = \delta(\rho).$$

Generalization in a higher dimension setting.

We still take $\Gamma = \pi_1(S_g)$. Let

$$\rho_1 : \Gamma \xrightarrow{\eta_0} \mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \left\{ \begin{bmatrix} * & * & \\ & * & \\ & & 1 \end{bmatrix} \right\} \subset \mathrm{SL}_3(\mathbb{R}).$$

We consider the action of $\mathrm{SL}_3(\mathbb{R})$ on $\mathbb{P}(\mathbb{R}^3) \cong \mathbb{S}^2$, which is non-conformal. For $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$ near ρ_0, we can define similarly the limit sets, which is a topological circle (ρ is in the Barbot component of $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$).

Theorem 1.4 (Barbot)

Assume that ρ is a small perturbation of ρ_1 which is irreducible. Then $L(\rho(\Gamma))$ is not Lipschitz.

Theorem 1.5 (Li-Pan-Xu)

Given any $\varepsilon > 0$, there exists an open neighborhood of ρ_1 in $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$ such that we have

- either ρ acts on $\mathbb{P}(\mathbb{R}^3)$ reducibly,
- or ρ acts irreducibly and

$$\left| \dim_{\mathrm{H}} L(\rho(\Gamma)) - \frac{3}{2} \right| < \varepsilon.$$

Now we define the affinity exponent of $\rho(\Gamma)$, $s_{\mathrm{A}}(\rho)$. We consider the Poincaré series

$$P_\rho(s) = \begin{cases} \displaystyle\sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)^s (\rho(\gamma)), & 0 < s \leqslant 1; \\ \displaystyle\sum_{\gamma \in \Gamma} \left(\frac{\sigma_2}{\sigma_1}\right)(\rho(\gamma)) \left(\frac{\sigma_3}{\sigma_1}\right)^{s-1} (\rho(\gamma)), & 1 < s \leqslant 2. \end{cases}$$

The **affinity exponent** is defined to be $s_{\mathrm{A}}(\rho) := \min\{ s > 0 : P_\rho(s) < \infty \}$. The concept of affinity exponent was first introduced by Falconer to study the Hausdorff dimension of self-affine fractals. Later it was generalized to different settings.

Recall. (X, d) a metric space and $S \subset X$. For every $s \geqslant 0$ and $\delta > 0$, we have

$$\mathscr{H}^s_\delta(S) = \inf \left\{ \sum \mathrm{diam}(U_i)^s : S \subset \bigcup U_i, \mathrm{diam}(U_i) < \delta \right\}.$$

Let $\mathscr{H}^s(S) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(S)$ and $\dim_{\mathrm{H}}(S) = \inf\{ s \geqslant 0 : \mathscr{H}^s(S) = 0 \}$.

Now we explain the intuition to the affinity exponent. To cover $L(\rho)$, we consider the image of a unit ball on $\mathbb{RP}^2$ by $\rho(\gamma)$. This is an ellipse with two axes of lengths σ_2/σ_1 and σ_3/σ_1. We can cover this ellipse by two ways: use a ball of radius σ_2/σ_1 or use σ_2/σ_3 balls of radius σ_3/σ_1. If such ellipses is not too much, the first way is more optimal. This corresponds to the case $s \leqslant 1$, where $P_\rho(s) = \sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}\right)^s(\rho(\gamma))$. For the case when there are much ellipses, we use the second way to cover each ellipse. This gives the expression of series for $s > 1$ as $\sum_{\gamma\in\Gamma}\left(\frac{\sigma_2}{\sigma_1}\right)(\rho(\gamma))\left(\frac{\sigma_3}{\sigma_1}\right)^{s-1}(\rho(\gamma))$.

Anosov representations in $\mathrm{SL}_3(\mathbb{R})$.

Definition 1.6. Let Γ be a hyperbolic group. Then $\rho : \Gamma \to \mathrm{SL}_3(\mathbb{R})$ is called **Anosov** if if there exists $c > 0$ such that for every $\gamma \in \Gamma$,

$$\frac{\sigma_1(\rho(\gamma))}{\sigma_2(\rho(\gamma))} > ce^{c|\gamma|},$$

where $|\gamma|$ is the word length of γ with respect to a fixed symmetric generating set.

Let $\mathrm{HA}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$ be the set of all Anosov representations from Γ to $\mathrm{SL}_3(\mathbb{R})$, which is an open subset of $\mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$.

Example 1.7 ρ_1 is Anosov, and hence its small perturbations are Anosov.

Theorem 1.8 (Li-Pan-Xu)
Let Γ be a hyperbolic group and $\rho : \Gamma \to \mathrm{SL}_3(\mathbb{R})$ be a Zariski dense Anosov proposition. Then

$$\dim_{\mathrm{H}} L(\rho(\Gamma)) = s_{\mathrm{A}}(\rho).$$

Moreover, $s_{\mathrm{A}}(\rho)$ is continuous with respect to ρ.

Some previous works on the Hausdorff dimensions of the limit sets of Anosov representations include:

- Pozzetti-Sambarino-Wienhard: $\dim_{\mathrm{H}} L(\rho(\Gamma)) \leqslant s_{\mathrm{A}}(\rho)$;
- Labourie, Benoist: if Γ a surface group and $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_3(\mathbb{R}))$ in the Hitchin component then $L(\rho(\Gamma))$ is C^1-circle;
- Glorieux-Monclair-Tholozan 19: projective Anosov representations;
- Dey-Kapovich 22, Dey-Kim-Oh 24: study the Hausdorff dimensions using metrics coming from other linear forms on the $\mathfrak{a}^+$.

Dimension formula for stationary measures.

Let (X, d) be a metric space and μ be a Borel probability measure on X. There are several notions of the dimension of μ:

- The Hausdorff dimension of μ is $\dim_{\mathrm{H}} \mu := \inf_{A\subset X, \mu(A)=1} \dim_{\mathrm{H}} A$.
- μ is called exact dimensional if for μ-a.e. x we have

$$\log_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

exists and equals to a constant. The limit is called the **exact dimension** of μ. By the work of Young, if μ is exact dimensional then the exact dimension of μ equals $\dim_{\mathrm{H}} \mu$. Moreover, we can compute $\dim \mu$ by entropy (see (2.1)).

- Let ν be a finitely supported on $\mathrm{SL}_3(\mathbb{R})$ with $\langle \operatorname{supp} \mu \rangle$ Zariski dense. Then ν admits a unique stationary measure μ on $\mathbb{P}(\mathbb{R}^3)$. Let $\lambda_1(\nu) > \lambda_2(\nu) > \lambda_3(\nu)$ be Lyapunov exponents of ν. The **Lyapunov dimension** of μ is defined as follows:

$$\dim_{\mathrm{LY}} \mu := \begin{cases} \dfrac{h_{\mathrm{F}}(\mu,\nu)}{\lambda_1(\nu)-\lambda_2(\nu)}, & \text{if } h_{\mathrm{F}}(\mu,\nu) \leqslant \lambda_1(\nu)-\lambda_2(\nu); \\ 1 + \dfrac{h_{\mathrm{F}}(\mu,\nu)-(\lambda_1(\nu)-\lambda_2(\nu))}{\lambda_1(\nu)-\lambda_3(\nu)}, & \text{otherwise.} \end{cases}$$

Theorem 1.9 (Li-Pan-Xu)

If ν is finitely supported, $\langle \operatorname{supp} \nu \rangle$ is Zariski dense and exponential separation (there exists $C > 0$ such that for every $x \neq y \in \operatorname{supp} \nu^{*n}$, $d(x,y) \geqslant e^{-Cn}$), then

$$\dim_{\mathrm{H}} \mu = \dim_{\mathrm{LY}} \mu.$$

§2 Lecture 2: Entropy growth argument

This time, we will explain the entropy growth argument, which is based on Hochman's work on Bernoulli convolutions.

Example 2.1 (Warm up: the standard $1/3$-Cantor set)

Let C_3 be the standard $1/3$-Cantor set. Let $\mu_{1/3}$ be the Cantor measure on C_3. We aim to compute the Hausdorff dimension of $\mu_{1/3}$. The approach is considering the exact dimension of $\mu_{1/3}$. For $r = (1/3)^n$, we have $\mu(B(x,r)) \approx (1/2)^n$ for almost every x. Therefore, $\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \log 2/\log 3$ for almost every x. This gives dimension of $\mu_{1/3}$. Moreover, we can note that $\log 2$ corresponds to the **entropy** and $\log 3$ corresponds to the **Lyapunov exponent**. The dimension is the quotient of these two quantities.

Bernoulli convolution.

Let $0 < \lambda < 1$. We consider two matrices

$$A_1 = \begin{bmatrix} \lambda & 1 \\ & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \lambda & -1 \\ & 1 \end{bmatrix}.$$

Let $\nu = \frac{1}{2}\delta_{A_1} + \frac{1}{2}\delta_{A_2}$ be the probability measure on $\mathrm{GL}_2(\mathbb{R})$.

These two matrices induce actions on the real line. That is, $A_1 x = \lambda x + 1$ and $A_2 x = \lambda x - 1$ for $x \in \mathbb{R}$. There exists a unique **stationary measure** μ_λ on $\mathbb{R}$ such that

$$\mu_\lambda = \nu * \mu_\lambda = \frac{1}{2}(A_1)_* \mu_\lambda + \frac{1}{2}(A_2)_* \mu_\lambda.$$

The measure μ_λ is called **Bernoulli convolution**. One can notice that μ_λ is supported on $I_\lambda = [-1/(1-\lambda), 1/(1-\lambda)]$.

For $\lambda < 1/2$, the matrices A_1, A_2 satisfy the separation condition. In this case, it is similar to the Cantor case and not hard to compute the dimension.

Question 2.2 For $\lambda > 1/2$, what is the dimension $\dim_{\mathrm{H}} \mu_\lambda$?

Conjecture 2.3 (Erdös)
μ_λ is absolutely continuous if $\lambda > 1/2$ and $1/\lambda$ is not Pisot.

Remark 2.4 An algebraic integer α is called **Pisot** if $\alpha > 1$ and all its Galois conjugates with absolute value < 1.

Erdös also showed that $\widehat{\mu}_\lambda(k) \not\to 0$ as $|k| \to \infty$ if λ^{-1} is Pisot. In particular, μ_λ is not absolutely continuous in the case. In fact, Garsia showed that $\dim_{\mathrm{H}} \mu_\lambda < 1$ in this case.

Theorem 2.5 (Hochman)
If λ is an algebraic number then $\dim \mu_\lambda = \min \{ 1, -h_\lambda / \log \lambda \}$.

Here h_λ is the **Garsia entropy** given by

$$h_\lambda := h_{\mathrm{RW}}(\nu) := \frac{1}{n} \lim_{n\to\infty} H(\nu^{*n}),$$

and H is the **Shannon entropy**. For the case $\lambda < 1/2$, the theorem covers the classical dimension computations. But this theorem also considered the case $\lambda > 1/2$ where there are some overlapping between the images of A_1 and A_2.

Recall. By a result of Feng-Hu, μ_λ is exact dimensional. By the work of Young, the dimension of μ_λ can be computed by the entropy:

$$\dim_{\mathrm{H}} \mu_\lambda = \lim_{n\to+\infty} \frac{1}{n} H(\mu_\lambda, \vartheta_n). \tag{2.1}$$

Here ϑ_n is the dyadic partition $\left\{ [\frac{k}{2^n}, \frac{k+1}{2^n}[: k \in \mathbb{Z} \right\}$ and

$$H(\mu, \vartheta_n) := \sum_{I\in\vartheta_n} -\mu(I) \log \mu(I).$$

Here the logarithm is taken in base 2. The idea to show (2.1) is using the exact dimensionality. Note that $\mu(I) \approx (1/2)^{n \dim \mu}$. We have

$$\sum -\mu(I) \log \mu(I) \approx \sum -\mu(I) n \dim \mu \log(1/2) = n \dim \mu.$$

Now we explain the idea of showing dimension formula in Theorem 2.5. To study μ_λ, the only thing we can use is the definition of stationary measures: $\mu_\lambda = \nu * \mu_\lambda = \nu^{*n} * \mu_\lambda$. Here ν^{*n} is supported on

$$\left\{ \begin{bmatrix} \lambda^n & \pm 1 \pm \lambda \pm \cdots \pm \lambda^{n-1} \\ 0 & 1 \end{bmatrix} \right\}.$$

We take $n' = [\log(1/\lambda) n]$ and q a positive integer large enough.

Definition 2.6. For integers $m > n$, we define the **conditional entropy** as

$$H(\mu, \vartheta_m | \vartheta_n) = H(\mu, \vartheta_m) - H(\mu, \vartheta_n) = \sum_{I\in\vartheta_n} \mu(I) H(\mu_I, \vartheta_m),$$

where $\mu_I = \frac{1}{\mu(I)} \mu|_I$.

Now we consider

$$\frac{1}{qn}H(\mu,\vartheta_{qn+n'}|\vartheta_{n'}) = \frac{1}{qn}H(\mu,\vartheta_{qn+n'}) - \frac{1}{qn}H(\mu,\vartheta_{n'}). \tag{2.2}$$

Letting $n \to \infty$, the limit is $\dim \mu$ by (2.1). Now we compute (2.2) in another way. We have

$$\frac{1}{qn}H(\mu,\vartheta_{qn+n'}|\vartheta_{n'}) = \frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H(\mu_I,\vartheta_{qn+n'}).$$

By the identity $\mu_\lambda = \nu^{*n} * \mu_\lambda$, we have $\mu_I \approx \nu_I^{*n} * \mu$. Here

$$\nu_I^{*n} := \frac{1}{\nu^{*n}(\varphi^{-1}(I))}\nu^{*n}|_{\varphi^{-1}(I)},$$

where $\varphi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}$ is given by $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mapsto b$. Hence we have

$$\frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H(\mu_I,\vartheta_{qn+n'}) = \frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H(\nu_I^{*n} * \mu,\vartheta_{qn+n'}). \tag{2.3}$$

Let $S_r : x \mapsto rx$ be the scaling map on $\mathbb{R}$ and $\boxplus$ be the additive convolution on $\mathbb{R}$. Then

$$\nu_I^{*n} * \mu = (\nu_I^{*n} * \delta_0) \boxplus S_{\lambda^n}\mu.$$

Now we estimate the lower bound of (2.3). The trivial bound is given as below

$$\begin{aligned}\frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H((\nu_I^{*n} * \delta_0)\boxplus S_{\lambda^n}\mu,\vartheta_{qn+n'}) &\geqslant \frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H(S_{\lambda^n}\mu,\vartheta_{qn+n'}) \qquad (2.4)\\ &= \frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H(\mu,\vartheta_{qn}). \qquad (2.5)\end{aligned}$$

Letting $n \to \infty$, the limit is also $\dim \mu$. Note that the trivial bound coincides with the actual value we computed before. This requires that there is no entropy growth in the additive convolution in (2.4).

Theorem 2.7 (Hochman)

For every $\varepsilon > 0$, C large enough, there exists $\delta > 0$ such that for every η_1, η_2 on $\mathbb{R}$ satisfying

(1) $\operatorname{diam}\operatorname{supp}\eta_1, \operatorname{diam}\operatorname{supp}\eta_2 \leqslant C2^{-k}$.
(2) $\frac{1}{n}H(\eta_1,\vartheta_{n+k}) > \varepsilon$,
(3) η_2 is ε-entropy porous.

Then $\frac{1}{n}H(\eta_1 \boxplus \eta_2,\vartheta_{n+k}) \geqslant \frac{1}{n}H(\eta_2,\vartheta_{n+k}) + \delta$.

To apply this entropy growth theorem with $\eta_1 = \nu_I^{*n} * \delta_0$ and $\eta_2 = S_{\lambda^n}\mu$, we need to verify the positivity of entropy of $\nu_I^{*n} * \delta_0$.

Why positive entropy of $\nu_I^{*n} * \delta_0$? Here we will take $\eta_1 = \nu_I^{*n} * \delta_0$ to obtain an entropy growth. The positivity of $H(\nu_I^{*n} * \delta_0)$ comes from the exponential separation and the contradiction hypothesis. Assume that $\dim \mu < \min\{1, -h_\lambda / \log \lambda\}$ then

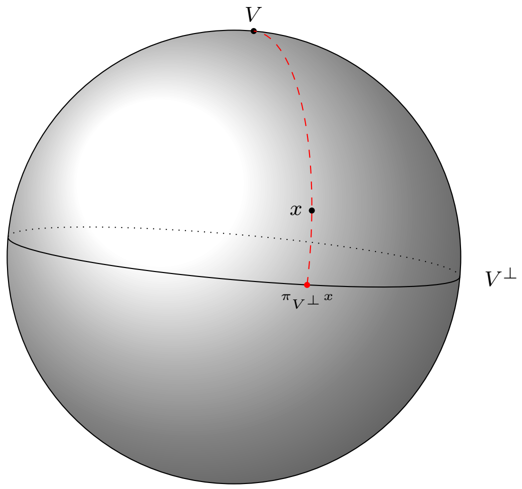
$$\begin{aligned}&\frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H(\nu_I^{*n}*\delta_0,\vartheta_{qn+n'}) = \frac{1}{qn}\sum_{I\in\vartheta_{n'}}\mu(I)H((\nu^{*n}*\delta_0)_I,\vartheta_{qn+n'})\\ =&\frac{1}{qn}H(\nu^{*n}*\delta_0,\vartheta_{qn+n'}|\vartheta_{n'}) = \frac{1}{qn}(H(\nu^{*n}*\delta_0,\vartheta_{qn+n'}) - H(\nu^{*n}*\delta_0,\vartheta_{n'}))\\ =&\frac{1}{qn}(H(\nu^{*n}) - H(\nu^{*n}*\delta_0,\vartheta_{n'})) \approx \frac{1}{qn}(nh_\lambda - n'\dim\mu) > 0.\end{aligned}$$

Here we use the exponential separation property to assert that $H(\nu^{*n}*\delta_0,\vartheta_{qn+n'}) = H(\nu^{*n})$ for some q large enough. The exponential separation property comes from the assumption that λ is algebraic.

§3 Lecture 3: Projection and non-concentration on arithmetic sequences

This lecture is devoted to explain some other key ingredients in the proof of Theorem 1.9.

Definition 3.1. Take $V \in \mathbb{P}(\mathbb{R}^3)$. Let $V^\perp \subset \mathbb{P}(\mathbb{R}^3)$ be the large circle corresponds to the orthogonal complement of V. We define the orthogonal projection $\pi_{V^\perp}x = V^\perp \cap \langle x, V\rangle$, where $\langle x, V\rangle$ is the large circle generated by x, V.



Let ν^- be the probability measure on $\mathrm{SL}_3(\mathbb{R})$ given by $\nu^-(g) = \nu(g^{-1})$. Let μ^- be the unique ν^--stationary measure on $\mathbb{P}(\mathbb{R}^3)$.

Theorem 3.2

Under the same condition. For μ^--almost every V, we have

$$\dim_{\mathrm{H}} \pi_{V^\perp}\mu = \min\left\{1, \frac{h_{\mathrm{RW}}(\nu)}{\lambda_1 - \lambda_2}\right\},$$

where $h_{\mathrm{RW}}(\nu) := \lim_{n\to\infty}\frac{1}{n}H(\nu^{*n})$.

Ledrappier-Young formula (by Ledrappier-Lessa, Rapaport). There exist γ_1, γ_2 such that

(1) $h_{\mathrm{F}}(\mu, \nu) = \gamma_1(\lambda_1 - \lambda_2) + \gamma_2(\lambda_1 - \lambda_3)$;
(2) $\dim_{\mathrm{H}} \mu = \gamma_1 + \gamma_2$;
(3) For μ^--almost every V, $\pi_{V^\perp}\mu$ is exact dimensional and $\dim \pi_{V^\perp}\mu = \gamma_1$;
(4) For μ^--almost every V, $\pi_{V^\perp}\mu$-almost every x, μ_x^V is exact dimensional and $\dim \mu_x^V = \gamma_2$, where μ_x^V is the measure along the fiber $\langle x, V\rangle$ that satisfies $\int \mu_x^V \,\mathrm{d}\pi_{V^\perp}\mu(x) = \mu$.

Now we show Theorem 1.9 by Theorem 3.2 and the Ledrappier-Young formula. This distinguishes two cases.

Case 1. If $h_{\mathrm{RW}}(\nu) < \lambda_1 - \lambda_2$ then we have

$$\frac{h_{\mathrm{RW}}(\nu)}{\lambda_1 - \lambda_2} = \dim \pi_{V^\perp}\mu \leqslant \frac{h_{\mathrm{F}}(\mu,\nu)}{\lambda_1 - \lambda_2} \leqslant \frac{h_{\mathrm{RW}}(\nu)}{\lambda_1 - \lambda_2}.$$

This first inequality is due to (1). Therefore, we have $h_{\mathrm{RW}}(\nu) = h_{\mathrm{RW}}(\mu, \nu)$, which implies the dimension formula.

Case 2. If $h_{\mathrm{RW}}(\nu) \geqslant \lambda_1 - \lambda_2$ then $\gamma_1 = \dim \pi_{V^\perp}\mu = 1$. Combining with (1) of Ledrappier-Young formula, we obtain the dimension formula. □

Non-concentration on arithmetic sequences. Another key ingredient to establishing the dimension formula is verifying the porosity condition for measures in Theorem 2.5. This needs the following "non-concentration on arithmetic sequences" property for measures.

Definition 3.3. Let μ be a probability measure on $\mathbb{R}/\mathbb{Z}$. We say μ satisfies **non-concentration on arithmetic sequences (NCAS)** if for every $\delta > 0$, there exists $k_0, \ell \geqslant 1$ such that for every $k \geqslant k_0$ we have

$$\mu\left(\bigcup_{0\leqslant n\leqslant 2^k} B\left(\frac{n}{2^k}, \frac{1}{2^{k+\ell}}\right)\right) < \delta.$$

Lemma 3.4

If μ satisfies the decaying property, i.e. there exists $0 < \varepsilon < 1$, $r_0 > 0$ and $C > 1$ such that for every $r < r_0$, $x \in \mathbb{R}/\mathbb{Z}$ $\mu(B(x, r/C)) \leqslant \varepsilon\mu(B(x,r))$, then μ is NCAS.

The difficulty is that our system is not uniformly contracting. The decaying property is hard to prove.

Definition 3.5. Let μ be a probability measure on $\mathbb{R}/\mathbb{Z}$. We call μ a **Rajchman measure** if

$$\widehat{\mu}(k) \to 0, \quad |k| \to +\infty,$$

where $\widehat{\mu}(k) = \int e^{2\pi i x k}\,\mathrm{d}\mu(x)$.

Proposition 3.6 If μ is Rajchman then μ satisfies NCAS.

Proof. Let f be an C^∞-bump function on $\mathbb{R}/\mathbb{Z}$ such that

$$f|_{[-2^{-\ell}, 2^{-\ell}]} = 1, \quad \operatorname{supp} f \subset [2^{-\ell+1}, 2^{-\ell+1}].$$

Let $F(x) = f(2^k x)$. Then

$$F|_{\bigcup_{0 \leqslant n \leqslant 2^k} B\left(\frac{n}{2^k}, \frac{1}{2^{k+\ell}}\right)} = 1$$

and hence

$$\mu\left(\bigcup_{0 \leqslant n \leqslant 2^k} B\left(\frac{n}{2^k}, \frac{1}{2^{k+\ell}}\right)\right) \leqslant \int F \, \mathrm{d}\mu.$$

Besides, we have

$$\int F \, \mathrm{d}\mu = \sum_{\xi \in \mathbb{Z}} \widehat{F}(\xi)\widehat{\mu}(-\xi).$$

The Fourier transform of F can be estimated as

$$\begin{aligned}
\widehat{F}(\xi) &= \int F(x) e^{2\pi i \xi x} \, \mathrm{d}x = \int f(2^k x) e^{2\pi i \xi x} \, \mathrm{d}x \\
&= \frac{1}{2^k} \int f(y) \sum_{0 \leqslant n < 2^k} e^{2\pi i \xi (y+n)/2^k} \, \mathrm{d}y \\
&= \int f(y) e^{2\pi i j y} \, \mathrm{d}y = \widehat{f}(j), \quad \text{where } \xi = 2^k j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left| \int F \, \mathrm{d}\mu \right| &= \left| \sum_j \widehat{f}(j) \widehat{\mu}(-2^k j) \right| \\
&\leqslant \widehat{f}(0)\widehat{\mu}(0) + \left(\sum_{j \neq 0} |\widehat{f}(j)| \right) \sup_{j \neq 0} \widehat{\mu}(-2^k j) \\
&\leqslant 2^{-(\ell-1)} + c(k) \|f\|_{C^2},
\end{aligned}$$

where $c(k) \to 0$ as $k \to \infty$ by the Rajchman property. □

How to show $\pi_{V^\perp}\mu$ is Rajchman?

Definition 3.7. The **flag variety** is $\mathscr{F} = \{ (V, W) : V < W < \mathbb{R}^3, \dim V = 1, \dim W = 2 \}$.

Theorem 3.8 (Jialun Li)

Let ν be a probability measure on $\mathrm{SL}_3(\mathbb{R})$. Assume that $\langle \operatorname{supp} \nu \rangle$ is Zariski dense and ν has finite exponential moment. There exist $\varepsilon_0, \varepsilon_1 > 0$ such that for all ξ large and for every $\varphi \in C^{1+\alpha}(\mathbb{P}(\mathbb{R}^3))$, $\gamma \in C^\alpha(\mathbb{P}(\mathbb{R}^3))$ that satisfy

(1) $\|\varphi\|_{C^{1+\alpha}} + \|\gamma\|_{C^\alpha} \leqslant \xi^{\varepsilon_0}$;

(2) $\mu_{\mathcal{F}} \{ (x, x \oplus v) : |(\mathrm{d}\varphi)_x(v)| < \xi^{-\varepsilon_0} \} < \xi^{-\varepsilon_0 \kappa}$ for some $\kappa > 0$, where $x \perp v$ are unit vectors.

Then

$$\left| \int e^{\xi \varphi(x)} r(x) \, \mathrm{d}\mu(x) \right| \leqslant \xi^{-\varepsilon_1} + \xi^{-\varepsilon_0 \kappa}.$$

Intuitively, the decay comes from the oscillation of φ, which at x is given by $\varphi(y) - \varphi(x) \approx (\mathrm{d}\varphi)_x((y-x))$. Note that x, y are in the limit set. In the non-conformal case, the distribution of $y - x$ may concentrate on a subspace of $T_x\mathbb{P}(\mathbb{R}^3)$. That is reason to consider the flag variety in (2).

In our case, we will take $\varphi(x) = \pi_{V^\perp}(x)$. Then $(\mathrm{d}\varphi)_x(v) = 0$ implies $v \in V$. Therefore, (2) follows from the large deviation on subvarieties.

§4 Lecture 4: Variational principle for Anosov representations

Recall our main theorem, Theorem 1.9. To show the dimension formula in Theorem 1.9, we use the following inequality.

$$s_{\mathrm{A}}(\rho) \geqslant \dim_{\mathrm{H}} L(\rho(\Gamma)) \geqslant \sup \dim_{\mathrm{H}} \mu = \sup \dim_{\mathrm{LY}} \mu \geqslant s_{\mathrm{A}}(\rho),$$

where μ is taken over all stationary measures induced by some nice random walks on $\rho(\Gamma)$. Here, the first inequality is established by Pozzetti-Sambarino-Wienhard. The third inequality is dimension formula for stationary measures by LPX. The last inequality is the following variational principle for affinity exponents.

Theorem 4.1 (Jiao-Li-Pan-Xu)

For every $\varepsilon > 0$, there exists a finitely supported probability measure ν on $\rho(\Gamma)$ with $\langle \operatorname{supp} \nu \rangle$ Zariski dense in $\mathrm{SL}_3(\mathbb{R})$ such that its unique stationary measure μ on $\mathbb{P}(\mathbb{R}^3)$ satisfies

$$\dim_{\mathrm{LY}} \mu \geqslant s_{\mathrm{A}}(\rho) - \varepsilon.$$

Now we fix a Zariski dense Anosov representation $\rho : \Gamma \to \mathrm{SL}_3(\mathbb{R})$. Recall that the definition of Lyapunov dimension involves the Furstenberg entropy

$$h_{\mathrm{F}}(\mu, \nu) = \int \log \frac{\mathrm{d} g\mu}{\mathrm{d}\mu}(\xi) \left(\frac{\mathrm{d} g\mu}{\mathrm{d}\mu}(\xi) \right) \mathrm{d}\nu(g) \mathrm{d}\mu(\xi).$$

However, this quantity is hard to compute. So we make use of the random walk entropy for later estimates.

Proposition 4.2

Let ν be a finitely supported probability measure on $\rho(\Gamma)$ with $\langle \operatorname{supp} \nu \rangle$ Zariski dense. Then its unique stationary measure on $\mathbb{P}(\mathbb{R}^3)$ satisfies

$$h_{\mathrm{F}}(\mu, \nu) = h_{\mathrm{RW}}(\nu).$$

Proof. We consider the $\mathrm{SL}_3(\mathbb{R})$ actions on $\mathbb{P}(\mathbb{R}^3)$ and $\mathcal{F}(\mathbb{R}^3)$, which admit the limit sets $L(\rho(\Gamma))$ and $L_{\mathcal{F}}(\rho(\Gamma))$ respectively. Let $\pi : \mathcal{F}(\mathbb{R}^3) \to \mathbb{P}(\mathbb{R}^3)$ is the projection, which is a factor map with respect to the $\mathrm{SL}_3(\mathbb{R})$-action.

$$\begin{tikzcd}
\mathrm{SL}_3(\mathbb{R}) \curvearrowright \mathbb{P}(\mathbb{R}^3) & L(\rho(\Gamma)) \arrow[l, hook'] \\
\mathrm{SL}_3(\mathbb{R}) \curvearrowright \mathcal{F}(\mathbb{R}^3) \arrow[u, "\pi"] & L_{\mathcal{F}}(\rho(\Gamma)) \arrow[l, hook'] \arrow[u, "\pi"]
\end{tikzcd}$$

By the monotonicity of Furstenberg entropy, we have

$$h_{\mathrm{RW}}(\nu) = h_{\mathrm{F}}(\mu_{\mathcal{F}}, \nu) \geqslant h_{\mathrm{F}}(\pi_* \mu_{\mathcal{F}}, \nu) = h_{\mathrm{F}}(\mu, \nu).$$

The reason of the first equality is that $\langle \operatorname{supp} \nu \rangle$ is discrete and hence $(\operatorname{supp} \mu_{\mathcal{F}}, \mu_{\mathcal{F}})$ is the Poisson boundary (Furman 02, Kaimanovich-Vershik, Ledrappier). Using the property of Anosov representations, the projection $\pi : L_{\mathcal{F}}(\rho(\Gamma)) \to L(\rho(\Gamma))$ has trivial fibers. Therefore, π is measure preserving, which gives $h_{\mathrm{F}}(\mu_{\mathcal{F}}, \nu) = h_{\mathrm{F}}(\mu, \nu)$. ☐

Now we state the key geometry input to show the variational principle.

Proposition 4.3 (Free sub-semigroups in hyperbolic groups)

There exists a finite subset $F \subset \Gamma$ with $\#F \geqslant 3$, constants $C_1, C_2, L_0 > 0$ and $m \in \mathbb{Z}_+$ such that the following holds.

For every subset $S \subset A(L)$ for some $L \geqslant L_0$ there exists a subset $S' \subset S$ with $\#S' \geqslant C_1^{-1}\#S$ and $F' \subset F$ with $\#F' = \#F - 2$ satisfying

(1) $\{\rho(f)^m : f \in F'\}$ generates Zariski dense semigroup in $\mathrm{SL}_3(\mathbb{R})$.

(2) $\widetilde{S} := \{sf^{\varsigma} : s \in S', f \in F', \varsigma = m, 2m\} \subset \Gamma$ freely generates a free semigroup.

(3) For any sequence of elements $\widetilde{s}_1, \cdots, \widetilde{s}_k \in \widetilde{S}$, we have

$$|\widetilde{s}_1 \cdots \widetilde{s}_k| \geqslant \sum_{i=1}^{k} |\widetilde{s}_i| - kC_2. \tag{4.1}$$

Here $A(L) := \{\gamma : |\gamma| = L\}$.

We now explain how this proposition deduce the variational principle.

Let $\mathfrak{a} = \{\lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) : \lambda_i \in \mathbb{R}, \sum_i \lambda_i = 0\}$ be a Cartan algebra of $\mathfrak{sl}_3(\mathbb{R})$ and $\mathfrak{a}^+ = \{\lambda \in \mathfrak{a} : \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3\}$ be a positive Weyl chamber. Set $A^+ = \exp \mathfrak{a}^+$ and $K = \mathrm{SO}_3(\mathbb{R})$. For every $g \in \mathrm{SL}_3(\mathbb{R})$, it admits the Cartan decomposition $g = \widetilde{k}_g a_g k_g \in KA^+K$. Here, $a_g = \mathrm{diag}(\sigma_1(g), \sigma_2(g), \sigma_3(g))$ where $\sigma_1(g) \geqslant \sigma_2(g) \geqslant \sigma_3(g)$ are singular values of g. The **Cartan projection** of g is defined to be

$$\kappa(g) := \mathrm{diag}(\log \sigma_1(g), \log \sigma_2(g), \log \sigma_3(g)) \in \mathfrak{a}.$$

A linear functional ψ on $\mathfrak{a}$ is called positive if

$$\psi = a_1\alpha_1 + a_2\alpha_2$$

with $a_1, a_2 \geqslant 0$ not all zero and $\alpha_1 = \lambda_1 - \lambda_2, \alpha_2 = \lambda_2 - \lambda_3$.

Proposition 4.4

Let ψ be a positive linear functional. If the series

$$\sum_{\gamma \in \Gamma} \exp(-\psi(\kappa(\rho(\gamma)))) \tag{4.2}$$

diverges, then there exists $c > 0$ such that the following holds. For every $\varepsilon > 0$, there exists infinitely many positive integers N with a finitely supported probability measure ν on $\rho(\Gamma)$ such that

(1) $\langle \mathrm{supp}\, \nu \rangle$ is Zariski dense in $\mathrm{SL}_3(\mathbb{R})$.

(2) $\lambda_p(\nu) - \lambda_{p+1}(\nu) \geqslant cN$ for every $p = 1, 2$.

(3) $h_{\mathrm{RW}}(\nu) \geqslant (1-\varepsilon)N$ and $\psi(\lambda(\nu)) \leqslant (1+\varepsilon)N$, where $\lambda(\nu) = (\lambda_1(\nu), \lambda_2(\nu), \lambda_3(\nu))$.

Proof. Applying Proposition 4.3, we obtain a finite subset $F \subset \Gamma$ with $\#F \geqslant 3$, constants $C_1, C_2, L_0 > 0$ and a positive integer m. Since (4.2) diverges, for every $\varepsilon > 0$ sufficiently small, there are infinitely many integers N such that

$$S_1 = \{\gamma \in \Gamma : \psi(\kappa(\rho(\gamma))) \leqslant N\}$$

has cardinality at least $e^{(1-\varepsilon)N}$. Since ψ is positive, there exists $c, c_1 \in (0,1)$ and $p = 1, 2$ such that

$$\psi(\kappa(\rho(\gamma))) \geqslant c_1(\log \sigma_p(\rho(\gamma)) - \log \sigma_{p+1}(\rho(\gamma))) \geqslant c|\gamma|.$$

Hence, $S_1 \subset \left\{\gamma \in \Gamma : |\gamma| \leqslant c^{-1}N\right\}$. Let $\mathcal{S}$ be a generating set of Γ and $c' = (2\log\#\mathcal{S})^{-1}$. Then

$$\#\left\{\gamma \in \Gamma : |\gamma| \leqslant c'N\right\} \leqslant 2(\#\mathcal{S})^{c'N} \leqslant e^{(1-\varepsilon)N}$$

for some small ε. Therefore, there exists $L \in [c'N, c^{-1}N]$ such that

$$S_2 = \{\, \gamma \in S_1 : |\gamma| = L \,\}$$

has cardinality at least $e^{(1-\varepsilon)N}/(c^{-1}N) \geqslant e^{(1-2\varepsilon)N}$. By Proposition 4.3, there exists $S_3 \subset S_2$ and $F' \subset F$ with $\#S_3 \geqslant C_1^{-1}\#S_2$ and $\#F' = \#F - 2$ such that

$$\widetilde{S} := \left\{sf^{\varsigma} : s \in S_3, f \in F', \varsigma = m, 2m\right\}$$

freely generates a free semigroup. We have $\#\widetilde{\mathcal{S}} \geqslant e^{(1-3\varepsilon)N}$. We take ν to be the uniform measure on $\rho(\widetilde{\mathcal{S}})$. Now we verify (1) – (3):

(1) We have $\rho(f)^m = \rho(sf^m)^{-1}\rho(sf^{2m})$. Hence $\rho(f)^m$ is contained in the Zariski closure of $\langle \operatorname{supp}\nu\rangle$. Hence $\langle \operatorname{supp}\nu\rangle$ is Zariski dense by (1) in 4.3.

(2) By the definition of Anosov representations, for every $p = 1, 2$ and $\gamma \in \Gamma$,

$$\log\sigma_p(\rho(\gamma)) - \log\sigma_{p+1}(\rho(\gamma)) \geqslant c|\gamma|.$$

By (4.1), for some $c > 0$ and every k, N large enough, we have $|\gamma| \geqslant ckN$ for every $\gamma \in \widetilde{\mathcal{S}}^{*k}$. Note that Lyapunov exponent can be given as the limit

$$\lambda_p(\nu) - \lambda_{p+1}(\nu) = \lim_{k\to\infty}\frac{1}{k}\int[\log\sigma_p(g) - \log\sigma_{p+1}(g)]\mathrm{d}\nu^{*k}(g).$$

We have $\lambda_p(\nu) - \lambda_{p+1}(\nu) \geqslant cN$.

(3) We have $h_{\mathrm{RW}}(\nu) = \log\#\widetilde{\mathcal{S}} \geqslant (1-3\varepsilon)N$. To estimate $\psi(\lambda(\nu))$, we need the following lemma:

> **Lemma 4.5** (Bochi-Potrie-Sambarino)
>
> Given an Anosov representation $\rho : \Gamma \to \mathrm{SL}_3(\mathbb{R})$. Let $p = 1, 2$. Then there exists $\delta > 0$ such that for every $\ell \leqslant k \leqslant m$, we have
>
> $$\sigma_p(\rho(\gamma_{\ell+1}\cdots\gamma_m)) \geqslant \delta\cdot\sigma_p(\rho(\gamma_{\ell+1}\cdots\gamma_k))\sigma_p(\rho(\gamma_{k+1}\cdots\gamma_m)),$$
>
> $$\sigma_{p+1}(\rho(\gamma_{\ell+1}\cdots\gamma_m)) \leqslant \delta^{-1}\cdot\sigma_{p+1}(\rho(\gamma_{\ell+1}\cdots\gamma_k))\sigma_{p+1}(\rho(\gamma_{k+1}\cdots\gamma_m)).$$

Hence we have

$$\left|\log\sigma_p(\rho(\widetilde{s}_1\cdots\widetilde{s}_k)) - \sum_{i=1}^{k}\log\sigma_p(\widetilde{s}_i)\right| \leqslant -k\log\delta$$

for every $\widetilde{s}_1, \cdots, \widetilde{s}_k \in \widetilde{S}$ and $1 \leqslant p \leqslant 3$. Since $\psi(\kappa(\rho(\widetilde{s}_i))) \leqslant N + C$ and ψ is linear, we have

$$\psi(\lambda(\nu)) = \lim_{k\to\infty}\frac{1}{k}\int\psi(\kappa(\rho(\gamma)))\mathrm{d}\nu^{*k}(\gamma) \leqslant (1+\varepsilon)N. \qquad \square.$$

Now we establish the variational principle by Proposition 4.4. For every $0 \leqslant s \leqslant 2$, let $\psi_s : \mathrm{SL}_3(\mathbb{R}) \to \mathbb{R}$ be the function given by

$$\begin{aligned}\psi_s(g) :=& \sum_{1\leqslant i\leqslant\lfloor s\rfloor}(\log\sigma_1(g) - \log\sigma_{i+1}(g)) + (s - \lfloor s\rfloor)(\log\sigma_1(g) - \log\sigma_{\lfloor s\rfloor+2}(g))\\ =& \inf\{a_{1,2}(\log\sigma_1(g) - \log\sigma_2(g)) + a_{1,3}(\log\sigma_1(g) - \log\sigma_3(g)) :\\ & \qquad 0 \leqslant a_{1,2}, a_{1,3} \leqslant 1, a_{1,2} + a_{1,3} = s\}.\end{aligned}$$

Then

$$s_A(\rho) = \sup \left\{ s : \sum_{\gamma \in \Gamma} \exp(-\psi_s(\rho(\gamma))) = \infty \right\}.$$

Apply Proposition 4.4 to ψ_s with $s = s_A(\rho) - \varepsilon$ and we obtain a probability measure ν on $\rho(\Gamma)$. Combining (2) and (3) in Proposition 4.4, we have

$$\psi_{s-\varepsilon'}(\lambda(\nu)) \leqslant (1-\varepsilon)N \leqslant h_{\mathrm{RW}}(\nu) = h_{\mathrm{F}}(\mu, \nu)$$

for $\varepsilon' = 2c^{-1}\varepsilon$. Observe that

$$\begin{aligned}\dim_{\mathrm{LY}} \mu &= \sup \{ a_{1,2} + a_{1,3} : 0 \leqslant a_{1,2}, a_{1,3} \leqslant 1, a_{1,2}(\lambda_1 - \lambda_2) + a_{1,3}(\lambda_1 - \lambda_3) = h_{\mathrm{F}}(\mu, \nu) \} \\ &= \sup \{ s : \psi_s(\lambda(\nu)) \leqslant h_{\mathrm{F}}(\mu, \nu) \}.\end{aligned}$$

We conclude that $\dim_{\mathrm{LY}}(\mu) \geqslant s - \varepsilon' \geqslant s_A(\rho) - 3\varepsilon'$.