Coding of the geodesic flow on geometrically finite hyperbolic 3-manifolds with a cusp

1 Coding of the geodesic flow and the limit set

Setup: Let Γ be a geometrically finite subgroup of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ such that $M := \Gamma\backslash\mathbb{H}^d$ (where $d = 2$ or 3) has only one cusp. Suppose $\infty \in \partial\mathbb{H}^d$ is a parabolic fixed point, and let $\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$ be the associated parabolic subgroup. We choose Δ_0 to be a fundamental domain for the action of Γ_∞ on $\partial\mathbb{H}^d \setminus \{\infty\} \simeq \mathbb{R}^{d-1}$. Let Λ be the limit set of Γ on $\partial\mathbb{H}^d$ and let $\Lambda_0 = \Lambda \cap \Delta_0$. Let μ be a Patterson-Sullivan measure supported on Δ_0. Let δ be the critical exponent.

Recall that we use the Hopf parametrization of the geodesic flow given by $(\partial\mathbb{H}^d \times \partial\mathbb{H}^d \setminus \mathrm{Diag}) \times \mathbb{R} \simeq T^1\mathbb{H}^d$. With an appropriate choice of identification, the action of Γ is given by

$$\gamma(x, y, t) = (\gamma x, \gamma y, t + \log |D\gamma(x)|), \tag{1.1}$$

where $\log |D\gamma(x)|$ denotes the conformal derivative, and the cocycle is cohomologous to the usual Busemann function.

We first explain how to obtain a coding of the geodesic flow using Proposition 1.1. The main coding result from [LP23] is the following:

Proposition 1.1. *There exists a family of disjoint open sets $\{\Delta_j\}_{j\geqslant 1}$ and an expanding map T on $\bigcup_j \Delta_j$ such that*

- $\mu(\Delta_0) = \sum_{j\geqslant 1} \mu(\Delta_j)$;
- *For each j, there exists $\gamma_j \in \Gamma$ such that $T|_{\Delta_j} = \gamma_j^{-1}$ and $\gamma_j(\Delta_0) = \Delta_j$;*
- *There exists $\lambda < 1$ such that for all j, $\|D\gamma_j\| \leqslant \lambda$ (where $\|D\gamma_j\|$ denotes the norm of the derivative with respect to the Euclidean metric on $\mathbb{R}^{d-1}$);*
- *(Exponential tail) There exists $\epsilon > 0$ such that*
$$\sum_{j\geqslant 1} \|D\gamma_j\|^{\delta-\epsilon} < \infty.$$

Indeed,

$$R(x) = \log \|D(\gamma_j^{-1})(x)\| \tag{1.2}$$

for $x \in \Delta_j$ is the return time, and the exponential tail condition can also be expressed as

$$\int_{\Delta_0} e^{\epsilon R(x)}\, d\mu(x) < \infty.$$

Before proceeding to the coding of the flow, we need to add a stable direction to obtain a section of the flow. This is achieved via the following proposition.

Proposition 1.2. *There exists a subset $\Lambda_- \subset \Lambda$ near ∞ such that for all j, we have $\gamma_j^{-1}(\Lambda_-) \subset \Lambda_-$.*

Under this identification, we let

$$S = \left(\bigcup_{j\geqslant 1} \Delta_j \right) \times \Lambda_- \times \{0\}$$

be a section of $T^1\mathbb{H}^d$, which also descends to a section in T^1M. Denote the projection map by π. **Note:** The coding system we construct is possibly larger than the geodesic flow itself, as we are not using the first return map to the section.

We construct a suspension flow on S by

$$S^R := S \times \mathbb{R}^+/(x, y, t + R(x)) \sim (Tx, T(x, y), t), \tag{1.3}$$

where for $x \in \Delta_j$, we have $Tx = \gamma_j^{-1}x \in \Delta_0$ and $T(x, y) = \gamma_j^{-1}y \in \Lambda_-$ by Proposition 1.2. On S^R we have a semiflow in the $\mathbb{R}^+$ direction, denoted by T_t.

The following lemma establishes the connection between the suspension flow and the geodesic flow. Here ν denotes a T-invariant measure absolutely continuous with respect to the Patterson-Sullivan measure (which exists by the general theory of expanding maps or Gibbs measures).

Lemma 1.3. *The map π induces a factor map from (S^R, T_t, ν^R) to $(T^1M, \mathcal{G}_t, m^{\mathrm{BMS}})$, that is*

$$\pi \circ T_t = \mathcal{G}_t \circ \pi, \quad \pi_*(\nu^R) = m^{\mathrm{BMS}}.$$

Proof sketch: Equations (1.3), (1.1) and (1.2) imply that π is well defined, since the equivalence relation is mapped to an equivalence relation under the Γ action. Indeed,

$$\gamma_j(Tx, \gamma_j^{-1}y, t) = \gamma_j(\gamma_j^{-1}x, \gamma_j^{-1}y, t) = (x, y, t + \log |D\gamma_j^{-1}(x)|) = (x, y, t + R(x)).$$

Since the geodesic flow is mixing, almost every geodesic trajectory will cross this transversal section $\pi(S)$ of T^1M.

The proof of the equality of measures uses that the T-invariant measure ν on Λ_0 is absolutely continuous with respect to the Patterson-Sullivan measure μ and the product structure; see [LP23, Prop. 4.15].

Remark 1.4. The subtle point here is that the map π from S^R to T^1M is not injective, since R may not be the first return time to the section. We only obtain a factor map, but this is still sufficient to deduce exponential mixing of the geodesic flow from that of the suspension flow; see [LP23, Section 4.2]. For the suspension flow, we use a modified Dolgopyat argument to prove its exponential mixing.

Since we do not require the first return to the section, the advantage is that the construction of the coding is more flexible. See also Young towers for a similar situation in the map case.

2 Limit set coding for $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$

We now construct the coding promised in Proposition 1.1. We work only on the boundary $\partial\mathbb{H}^d$. The idea is to use "flowers" near parabolic fixed points (see Figure 1). The advantage is that the boundary of a flower is clean, which enables us to obtain good measure estimates and thus the exponential tail property.

Let H_0 be the horoball based at infinity. Let $p = \gamma\infty$ be a parabolic fixed point in Δ_0 and let $H_p = \gamma H_0$ be the horoball based at p, for some $\gamma \in \Gamma$ with $\gamma^{-1}\infty \in \Delta_0$. Let h_p be the height of H_p, that is, the Euclidean height in the upper half-space model.

We tile a neighborhood of p using a flower J_p as follows. Let $\eta > 0$ be a small parameter. Let F_∞ be the complement of a large parallelotope of size $1/\eta$ in the tiling of $\mathbb{R}^{d-1}$ by Γ_∞-translates of Δ_0. Let

$$N_p = \{\gamma h \in \Gamma_\infty : h \in \Gamma_\infty, h\Delta_0 \notin F_\infty\}, \quad J_p = \gamma F_\infty^c = \bigcup \{\gamma h \Delta_0 : \gamma h \in N_p\}.$$

By computation, we obtain the following:

Lemma 2.1. *There exists $C > 0$ such that for any parabolic fixed point $p \in \Delta_0$, we have*

$$B_{\mathrm{Euc}}(p, \eta h_p / C) \subset J_p \subset B_{\mathrm{Euc}}(p, C\eta h_p),$$

where B_{Euc} denotes the Euclidean ball.

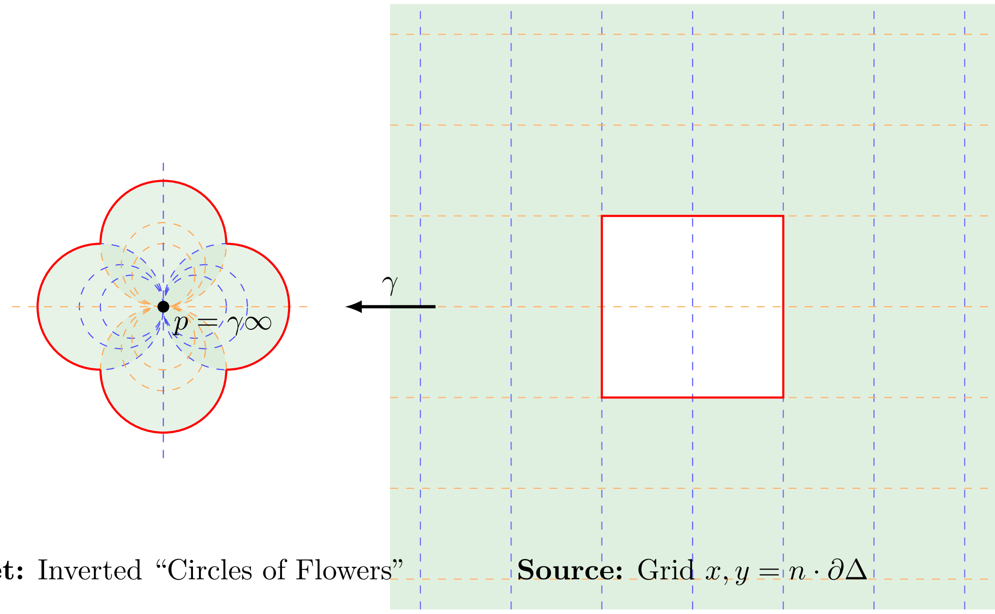

Figure 1: The shaded region on the left is J_p, which is the image under the action of γ of the complement of the white rectangle on the right.

Remark 2.2. For the non-maximal rank case, we need to modify the definition of J_p slightly. Since the direction without translation action of Γ_∞ is bounded, we take a large set to cover it.

In this way, we obtain a set J_p near a parabolic fixed point, which serves as a building block for the coding. In order to code almost all of the set Λ_0, we need to use flowers of decreasing size. However, we must ensure that the remaining set has good measure estimates, so we only select flowers that are sufficiently far from the boundary.

The procedure is as follows:

- Let $\Omega_0 = \Delta_0$.
- For $n \in \mathbb{N}$, let P_n be the set of parabolic fixed points in Ω_n such that $-\log(\eta h_p) \in [n, n+1]$ and $d(p, \partial\Omega_n) > h_p$. (Note that J_p has size roughly e^{-n}, and its distance to the boundary is of size e^{-n}/η.)
- Let
$$\Omega_{n+1} = \Omega_n \setminus \bigcup_{p \in P_n} J_p.$$

The key proposition is the following:

Proposition 2.3. *There exist constants $C' > 0$ and $\epsilon > 0$ such that*

$$\mu(\Omega_n) \leqslant C'(1-\epsilon)^n.$$

From this proposition, we obtain the coding sets by taking $\{\Delta_j\}_{j \geqslant 1} = \{\gamma\Delta_0 : \gamma \in N_p, p \in P_n, n \in \mathbb{N}\}$.

3 Mass exchange and exponential tail

This section is devoted to proving Proposition 2.3. We return to the flow and manifold point of view to obtain more information about these flowers.

For a point $x \in \Delta_0$, let $\tilde{x}$ be its lift to $T^1\mathbb{H}^d$ based at ∂H_0 whose forward endpoint is x. Let

$$C(\eta) = \bigcup_{p \text{ parabolic fixed}} T^1(H_p(\eta))$$

be the union of horoballs with height ηh_p, whose projection to T^1M is exactly the cusp region.

At time n, the set $\mathcal{G}_n(\tilde{\Omega}_n)$ is a large piece with many holes coming from "flowers" of different previous steps. For points in $\mathcal{G}_n(\tilde{\Omega}_n) \setminus C(\eta)$, we have uniform recurrence of the geodesic flow, which enables us to find a new flower near x (Lemma 3.3) in the next few steps. For the part $\mathcal{G}_n(\tilde{\Omega}_n) \cap C(\eta)$, we cannot use recurrence. This is the union of the intersection of $\mathcal{G}_n(\tilde{\Omega}_n)$ with different horoballs $H_p(\eta)$. We want to show that at the next step, a positive proportion of $\mathcal{G}_n(\tilde{\Omega}_n) \cap H_p(\eta)$ leaves the cusp $C(\eta)$ (equation (3.1)).

This motivates us to introduce a mass exchange argument inspired by the work of Lai-Sang Young on towers for maps.

Definition 3.1. For $n \in \mathbb{N}$:

- Let
$$D_n = \bigcup_{p \in P_n} J_p$$
be the union of flowers selected at step n.
- Let
$$B_n = \{x \in \Omega_n : \mathcal{G}_n(\tilde{x}) \in C(\eta)\}, \quad A_n = \Omega_n \setminus B_n.$$
The set B_n consists of points corresponding to parabolic fixed points that are too close to the boundary to be selected at step n. Hence these points correspond to regions in the cusps.
- Let
$$\Omega'_n = \{x \in \Omega_n : d(x, \partial\Omega_n) \leqslant e^{-n}\}$$
be the set of points in Ω_n within distance e^{-n} of the boundary.

Lemma 3.2.
- *For all $n \in \mathbb{N}$, we have*
$$B_n \cap D_{n+1} = \emptyset.$$
- *(Re-entry: mass flows from the cusp region to the non-cusp region) There exists $c_1 > 0$ such that*
$$\mu(B_n \cap A_{n+1}) \geqslant c_1 \mu(B_n). \tag{3.1}$$
- *(Escape: mass flows from the non-cusp region to the cusp region or the coded region) There exists $c_2 > 0$ (depending on η) such that*
$$\mu(A_n \cap (D_{n+1} \cup B_{n+1})) \leqslant c_2 \mu(A_n) + \mu(A_n \cap \cap B_{n+1} \cap \Omega'_n). \tag{3.2}$$

The following lemma follows from the uniform recurrence of the geodesic flow.

Lemma 3.3 (Recurrence)**.** *There exist $N \in \mathbb{N}$ and $c_0 > 0$ such that for any $n \in \mathbb{N}$,*

$$\mu\left(\bigcup_{1 \leqslant l \leqslant N} D_{n+l}\right) > c_0 \mu(A_n \setminus \Omega'_n).$$

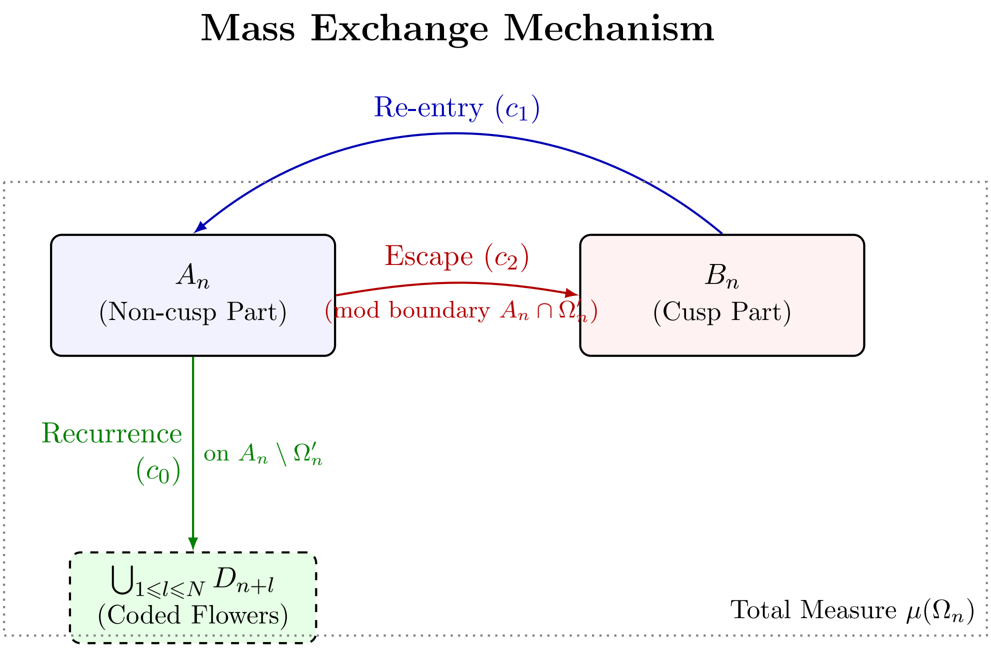

Figure 2: The mass exchange mechanism between the cusp region B_n and the non-cusp region A_n, with coding occurring on $A_n \setminus \Omega'_n$.

We also need a boundary estimate.

Lemma 3.4 (Boundary)**.** *There exists* $c_3 > 0$ *(tending to* 0 *as* $\eta \to 0$*) such that*

$$\mu(A_n \cap \Omega'_n) < c_3 \mu(\Omega_n).$$

We postpone the proofs of these lemmas and first explain the main mass exchange argument.

Proof of Proposition 2.3. It suffices to show that there exists $\epsilon > 0$ such that

$$\mu\left(\bigcup_{1 \leqslant l \leqslant N} D_{n+l}\right) > \epsilon \mu(\Omega_n).$$

Step 1: We obtain a uniform lower bound for the ratio $t_n = \mu(A_n)/\mu(B_n)$. By equations (3.2), (3.1), and Lemma 3.4,

$$\begin{aligned} t_{n+1} = \frac{\mu(A_{n+1})}{\mu(B_{n+1})} &= \frac{\mu(A_n) + \mu(B_n \cap A_{n+1}) - \mu(A_n \cap (D_{n+1} \cup B_{n+1}))}{\mu(B_n) - \mu(B_n \cap A_{n+1}) + \mu(A_n \cap B_{n+1})} \\ &\geqslant \frac{\mu(A_n) + c_1 \mu(B_n) - (c_2 \mu(A_n) + c_3 \mu(\Omega_n))}{\mu(B_n) - c_1 \mu(B_n) + (c_2 \mu(A_n) + c_3 \mu(\Omega_n))} \\ &= f(t_n), \end{aligned}$$

where f is a fractional linear function. Since f is convex, we have

$$\inf_{t \in \mathbb{R}^+} f(t) \geqslant \min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\} = q(c_3),$$

where a_i, b_i are the coefficients of f. Since $t_n > 0$, there exists a uniform lower bound for t_n for all $n \in \mathbb{N}$.

Step 2: Using Lemma 3.3, we obtain the desired statement:

$$\mu\left(\bigcup_{l=1}^{N} D_{n+l}\right) \geqslant c_0\mu(A_n \setminus \Omega'_n) \geqslant c_0(\mu(A_n) - \mu(A_n \cap \Omega'_n))$$
$$\geqslant c_0(\mu(A_n) - c_3\mu(\Omega_n)) = c_0\left(\frac{t_n}{1+t_n} - c_3\right)\mu(\Omega_n).$$

□

Proof of Lemma 3.2.

- The set $B_n \setminus B_{n-1}$ is essentially the projection of the intersection of $\mathcal{G}_n(\tilde{\Omega}_n)$ with the union of new horoballs H_p with $-\log(\eta h_p) \in [n+1, n+2]$ but with $d(p, \partial\Omega_n) < h_p$. The set $B_{n+1} \cap B_n$ is the union of previous bad horoballs. By the separation property of horoballs, we obtain $B_n \cap D_{n+1} = \emptyset$.
- This estimate comes from $B_n \setminus B_{n+1}$. Decomposing into disjoint horoballs and using the regularity of the Patterson-Sullivan measure, we have

$$\mu(B(x, e^{-n/2})) - \mu(B(x, e^{-(n+1)/2})) \geqslant c_1\mu(B(x, e^{-n/2})).$$

  (The delicate part is that the connected component in $B_n \setminus B_{n+1}$ may be a portion of an annulus. We introduce equivalence classes to deal with this issue.)
- This follows from the separation between parabolic fixed points. For the second term, it follows from the fact the near boundary, we can only get points in bad set B_{n+1}.

□

Proof of Lemma 3.3. We use the uniform recurrence of the geodesic flow. For any compact subset K of the non-wandering set W, there exists a time U such that for any $y \in K$, the unstable leaf $\mathcal{G}_t(W^u(y, 1))$ meets $\partial C(\eta)$ for some $t \in [0, U]$. The proof uses the ergodicity of the geodesic flow.

Now take $z = \tilde{x} \in A_n \cap \Lambda \setminus \Omega'_n$ and let $y = \mathcal{G}_n(z)$. Then y lies in the compact set $W \setminus \Gamma C(\eta)$, the complement of the cusp region. By applying the recurrence property to $W^u(y, 1)$, we can find a suitable parabolic fixed point p near x. Since x is away from the boundary of Ω_n, this point p will be a good parabolic fixed point and we can take the flower J_p.

(Technical detail: The actual argument is more involved, since the recurrence only implies $m = [-\log(\eta h_p)] \in [n, n+U]$. When we are at step m, the point p may not be in P_m. However, we can analyze the reason why it is not in P_m to find another nearby parabolic fixed point.)

Finally, the conclusion follows from the doubling property of the measure. □

Proof of Lemma 3.4. The proof relies on the non-concentration property of the Patterson-Sullivan measure near boundaries.

We use the friendliness of the Patterson-Sullivan measure: There exist $\epsilon, c > 0$ such that for any $r \in [0, 1]$, any line l in $\mathbb{R}^2$, and any $x \in N_{\epsilon r}(l)$, there exists $\rho_x > 0$ such that

$$\mu(B(x, \rho_x) \cap (N_r(l) \setminus N_{\epsilon r}(l))) \geqslant c\,\mu(B(x, \rho_x)).$$

Once we have this estimate, by the Vitali covering lemma, we conclude that

$$\mu(N_{\epsilon r}(l)) \leqslant (1-c)\mu(N_r(l)).$$

We can iterate ot obtain a desired estimate. For the geometrically finite case, this friendliness property is due to Das-Fishman-Simmons-Urbanski.

□

References

[LP23] Jialun Li and Wenyu Pan. Exponential mixing of geodesic flows for geometrically finite hyperbolic manifolds with cusps. *Inventiones mathematicae*, 231(3):931–1021, March 2023.