Structural stability.

Key idea: chaos is stable under small perturbation

Thm 7.1 (Fixed pt). Let (X, d) be a complete metric space. and $\varphi: X \to X$ is δ-Lip function with $\delta < 1$. Then φ has a unique fixed pt ℓ. Moreover, for any $x \in X$,

$$d(x, \ell) \le (1-\delta)^{-1} d(x, \varphi(x))$$

Pf: x_0. $x_n = \varphi(x_{n-1})$, then

$$d(x_{n+1}, x_n) = d(\varphi^n(x_1), \varphi^n(x_0)) \le \delta^n d(x_1, x_0)$$

So $\{x_n\}_{n \in \mathbb{N}}$ a Cauchy sequence. $\exists$ unique limit ℓ.

$$d(\ell, \varphi(\ell)) = \lim d(x_n, \varphi(x_n)) = \lim d(x_n, x_{n+1}) = 0$$

$$d(x, \ell) = \lim d(x_0, x_n)$$

$$\le \sum \delta^n d(x_1, x_0) = (1-\delta) d(x, \varphi(x)) \quad \square$$

Application. Equip $\mathbb{R}^N$ with a norm. $\|\cdot\|$. For a linear map T. let operator norm

$$\|T\| = \sup_{\|x\|=1} \frac{\|Tx\|}{\|x\|}.$$

Lem 7.2 If T is a linear bijection on $\mathbb{R}^N$ and $f: \mathbb{R}^N \to \mathbb{R}^N$ is δ-Lip with $\delta \le \|T^{-1}\|^{-1}$. Then $F = T + f$ is a Lip homeo

Pf: $F = T \circ (\mathrm{Id} + T^{-1} f)$

T invertible Lip. it is sufficient to consider

$\mathrm{Id} + f$ with $\mathrm{Lip}(f) \leq \|T^{-1}\| \cdot \delta < 1$.

For x,
We want to find y s.t.

$y + f(y) = x, \qquad y = x - f(y)$

Let $\varphi(y) = x - f(y)$. then $\mathrm{Lip}\,\varphi = \mathrm{Lip}\, f < 1$

By Fixed pt, $\exists$ unique y. s.t. $\varphi(y) = y$

~~and d i~~ Then $F(y) = y + f(y) = x$.

F Lip. $\cancel{\|F^{-1}(y) - F^{-1}(x')\|}$

$\| F(x) - F(x') \| \geq \|x - x'\| - \|f(x) - f(x')\| \geq (1 - \mathrm{Lip} f) \|x - x'\|$

$\Rightarrow F^{-1}$ is $(1 - \mathrm{Lip} f)^{-1}$ Lip $\square$

Hyperbolic linear automorphism of $\mathbb{R}^N$

A linear automorphism T is called hyperbolic

if $\exists$ linear subspace E^s, E^u. s.t $\mathbb{R}^N = E^s \oplus E^u$

and norm $\|\cdot\|$. $T = S + U$. s.t $TE^{s/u} = E^{s/u}$

$T|_{E^s} = S, \quad T|_{E^u} = U$ and $\exists\, n \geq 1$

$\|S^n\| < 1, \quad \|U^{-n}\| < 1$

Rq: i) The definition doesn't depend on the choice of norm $\begin{pmatrix} * & \\ & * \end{pmatrix}$

ii) the norm is called adapted. if

$\|x\| = \max\{\|x^u\|, \|x^s\|\} \qquad \|S\|, \|U^{-1}\| < 1$.

Let ~~hyp~~ constant of hyperbolicity

$$ch(T) = \max\{\|S\|, \|U^{-1}\|\} < 1$$

depending on the choice of norm

__lem__: 7.3 Let T be a linear automorphism on $\mathbb{R}^N$. Then

T is hyperbolic $\Longleftrightarrow$ T has no eigenvalue of norm 1

In this case, there exists an adapted norm

Pf. $\Rightarrow$ v. $Tv = \lambda v$ eigenvector

$$p_T(X) = \prod_{|\lambda_j|<1}(x-\lambda_j) \prod_{|\lambda_j|>1}(x-\lambda_j).$$

$= P_1 P_2$

$\Leftarrow$ No eigenvalue of norm 1. ~~$\mathbb{R}^N = \ker$~~

$\mathbb{R}^N = \ker(P_1) \oplus \ker(P_2)$

$$T = P\,(\text{Jordan blocks})\,P^{-1} \qquad \left(\begin{array}{ccc|ccc} \lambda_1 & 1 & & & & \\ & \ddots & 1 & & & \\ & & \lambda_j & & & \\ \hline & & & \lambda_{j+1} & 1 & \\ & & & & \ddots & 1 \\ & & & & & \lambda_n \end{array}\right)$$

$E^s = \{v, |\text{eigenvalue}| < 1\}$ $\qquad E^u = \{v, |\text{eigenvalue}| > 1\}$

$T = S + U$. by $S^n \to 0$ $\quad U^{-n} \to 0$

~~we~~ $\exists\, n$. s.t. $\|S^n\| < 1$, $\|U^{-n}\| < 1$

Here $\|\cdot\|$ is any norm, we construct an adapted norm

$$\|x\|' = \max\left\{\sum_{k=0}^{n-1}\|S^k x^s\|, \sum_{k=0}^{n-1}\|U^k x^u\|\right\}.$$

Then $$\|S\|' = \sup_{\|x^s\|'=1} \frac{\|Sx^s\|'}{\|x^s\|'} = \sup \frac{\sum_{k=1}^{n}\|S^k x^s\|}{\sum_{k=0}^{n-1}\|S^k x^s\|}$$

$$\sum_{k=1}^{n}\|S^k x^s\| < \sum_{k=0}^{n-1}\|S^k x^s\|. \Longleftarrow$$

By compactly of $\|x^s\|' = 1$. we have $\|S\|' < 1$ $\square$

Small perturbation of an hyperbolic automorphism of $\mathbb{R}^N$ is topolgically conjugated to itself

Prop Let T be a linear hyperbolic automorphism of $\mathbb{R}^N = E$. We fix a norm $\|\cdot\|$ adapted to T. Let $F = T + f$ where f is a bounded and δ-Lip with $\delta < \inf\{1 - ch(T), \|T^{-1}\|^{-1}\}$. Then $\exists$ unique homeo $H = id + h$ s.t. h is uniformly continuous and bounded with

$$H \circ T = F \circ H.$$

Rq.
1) F has a unique fixed pt $H(0)$
3) h bounded used to make H unique (Hidden in $\mathcal{H}$)
2). We cannot make H Lip. ex. $E = \mathbb{R}$, $Tx = 2x$, $f(x) = -\varepsilon \sin x$.

$H(0) = 0$. $\quad x_n = 2^{-n} \to 0$, $\quad y_n = H(x_n) \to 0$

then
$$\frac{y_{n+1}}{y_n} = \frac{y_{n+1}}{F(y_{n+1})} \to \frac{1}{F'(0)} = \frac{1}{2-\varepsilon}$$

$$\begin{tikzcd}
x_{n+1} \quad E \arrow[r, "T"] \arrow[d, "H"'] & E \quad x_n \arrow[d, "H"] \\
y_{n+1} \quad E \arrow[r, "F"] & E \quad F(y_{n+1}) = y_n
\end{tikzcd}$$

~~But~~ Therefore
$$\frac{H(x_n)}{x_n} = \frac{y_n}{x_n} = \left(\prod_{k \le n} \frac{y_k}{y_{k-1}}\right) \Big/ \left(\prod_{k \le n} \frac{x_k}{x_{k-1}}\right) \xrightarrow{n \to \infty} \infty$$

$$\cancel{d(H(x_n), H(0)) / d(x_n, 0) \to \infty}$$

$$\mathcal{H} = UCB(E, E) = \{ f : E \to E, \text{ bounded uniformly continuous}, \|\cdot\|_\infty \}$$

Prop. 7.5 Let T be a linear bijection on $E = \mathbb{R}^N$. We fix a norm $\|\cdot\|$ adapted to T. Let $F = T + f$ and $G = T + g$, where f, g ~~two~~ bounded δ-Lip with $\delta < \inf\{1 - ch(T), \|T^{-1}\|^{-1}\}$.

i) $\exists$ unique $h \in \mathcal{H}$ s.t. $H = id + h$ satisfies

$$F \circ H = H \circ G, \qquad \|h\|_\infty \le (1 - \delta - chT)^{-1} \|f - g\|_\infty$$

(ii). H is ~~an~~ a homeo

Pf: i) For a map $\gamma : E \to E$ let

$\gamma_s : E \to E^s$ $\gamma_u = E \to E^u$.

Then $F \circ H = H \circ G$

$\Leftrightarrow T \circ H + f \circ H = G + h \circ G.$

$\Leftrightarrow \begin{cases} S \circ h_s + f_s \circ H = g_s + h_s \circ G \\ U \circ h_u + f_u \circ H = g_u + h_u \circ G \end{cases}$

$\Leftrightarrow$ (☆) $\tilde{h}_s = (S \circ h_s + f_s \circ H - g_s) \circ G^{-1} = \varphi_s(h_s)$ h_s fixed pt

$\tilde{h}_u = U^{-1} \circ (-f_u \circ H + g_u + h_u \circ G) = \varphi_u(h_u)$ h_u fixed pt

Want to use fixed pt ~~thm~~., By lem 7.2. $\delta < \|T^{-1}\|^{-1}$, G. bi-Lip

$$\|\tilde{h}_s - \tilde{h}_s'\|_\infty \le (\|S \circ h_s - S \circ h_s'\|_\infty + \|f_s \circ H - f_s \circ H'\|_\infty) \cancel{\|G\|}$$

$$\le (ch(T) + \delta)\|h_s - h_s'\|_\infty$$

$$\|\tilde{h}_u - \tilde{h}_u'\|_\infty \le \|U^{-1}(f_u \circ H - f_u \circ H')\|_\infty + \|U^{-1}(h_u - h_u')\|_\infty$$

$$\le (\delta + ch(T))\|h_u - h_u'\|_\infty$$

By fixed pt thm. $\exists !$ h_s. h_u satisfies (☆)

$$\|h_p\|_\infty = d(\varphi_p(h_p), \overset{0}{\cancel{\varphi_s(0)}}) \le (1 - \delta - chT)^{-1} d(\overset{\varphi(0)}{h_s}, 0)$$

$$= (1 - \delta - chT)^{-1} \|f - g\|_\infty.$$

(ii). We can solve H' from

$$H' \circ F = G \circ H',$$

then $(H \circ H') \circ F = H \circ G \circ H' = F(H \circ H')$. By unicity. $H \circ H' = id$

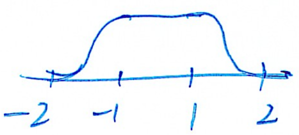
Local conjugate. Groβman - Hartmann

Thm 7.6: Let Ω be an open subset of $\mathbb{R}^N$. $\varphi : \Omega \to \mathbb{R}^n$ a C^1 local diffeo and x_0 a hyperbolic fixed pt of φ.
(i.e. $\varphi(x_0) = x_0$, $T = D\varphi(x_0)$ is hyperbolic automorphism)
Then $\exists$ neighborhood U at 0 and V at x_0 in Ω ~~st~~ and $H : U \to V$ homeo. s.t. $\forall\, x \in U$, $T(x) \in V$.

$$H \circ T(x) = \varphi \circ H(x)$$

Pf: Approximate φ by $T + \cancel{f}$ ~~at small~~ locally.

Let $\alpha(x) \in C^1$ $\qquad \alpha|_{B(0,1)} = 1$, $\quad \operatorname{supp} \alpha \subset B(0,2)$



Let $\alpha_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right)$. a small partition function

Let $f(x) = \varphi(x) - \left(T(x - x_0) + x_0\right)$ $\qquad f_\varepsilon = \cancel{\alpha_\varepsilon f}.\ \alpha_\varepsilon(x - x_0) f$

and $F_\varepsilon = T + f_\varepsilon$. Then $F_\varepsilon = \varphi$ on $B(x_0, \varepsilon)$.
$F_\varepsilon = T$ outside $B(x_0, 2\varepsilon)$

Want to apply Prop 7.4: f_ε bonded.

$$
\begin{aligned}
\operatorname{Lip}(f_\varepsilon) &\le \operatorname{Lip}(\alpha_\varepsilon) \cdot \|f\|_{\infty,\, B(x_0,\varepsilon)} + \|\alpha_\varepsilon\|_\infty \operatorname{Lip}(f|_{B(x_0,\varepsilon)}) \\
&\le \frac{1}{\varepsilon} \cdot \cancel{3\varepsilon}\, \| f|_{B(x_0,\varepsilon)} \|_\infty + \sup_{x \in B(x_0,\varepsilon)} \| Df(x) \| \\
&\le \|\alpha\|_{C^1} \sup_{x \in B(x_0,\varepsilon)} \| Df(x) \|.
\end{aligned}
$$

as $\varepsilon \to 0$ $\longrightarrow 0$

Hence we can use Prop 7.4. and find H_ε. s.t

$$H_\varepsilon \circ T = F_\varepsilon \circ H_\varepsilon. \quad \text{~~on~~}$$

On $B(x, \varepsilon)$. we obtain

$$H_\varepsilon \circ T = \varphi_\circ H_\varepsilon . \qquad \square$$

Structural stability of hyperbolic torus automorphism (Anosov).

$p: \mathbb{R}^N \xrightarrow{\text{lift}} \mathbb{T}^N$

lem: (Covering of loops). For all continuous $\varphi: [0,1] \to \mathbb{T}^N$

and $x_0 \in \mathbb{R}^N$ with $p(x_0) = \varphi(0)$.

Then $\exists$ unique continuous $\hat{\varphi}: [0,1] \to \mathbb{R}^N$ s.t.

$$\hat{\varphi}(0) = x_0 \quad \text{and} \quad p \circ \hat{\varphi} = \varphi.$$

lem: (lift of maps). For all continuous maps $\psi: \mathbb{R}^{N'} \to \mathbb{T}^N$. and $x_0 \in \mathbb{R}^N$ with $p(x_0) = \psi(0)$ Then $\exists !$ continuous $\hat{\psi}: \mathbb{R}^{N'} \to \mathbb{R}^N$. s.t. $\hat{\psi}(0) = x_0$, and $p \circ \hat{\psi} = \psi$.

$$\begin{tikzcd} & \mathbb{R}^N \ni x_0 \arrow[d, "p"] \\ \mathbb{R}^{N'} \arrow[ur, "\hat{\psi}"] \arrow[r, "\psi"] & \mathbb{T}^N \ni \psi(0) \end{tikzcd}$$

Cor: For all continuous $f: \mathbb{T}^N \to \mathbb{T}^N$

and $x_0 \in \mathbb{R}^N$ s.t. $p(x_0) = f(x_0)$.

1. $\exists !$ a continuous $\hat{f}: \mathbb{R}^N \to \mathbb{R}^N$, lift of f. s.t $\hat{f}(0) = x_0$, $p \circ \hat{f} = f \circ p$.

2. $\exists$ matrix f_*. with integer coefficients s.t.

$$\forall x \in \mathbb{R}^N, \quad z \in \mathbb{Z}^n \qquad \hat{f}(x+z) = \hat{f}(x) + f_*(z)$$

3. The matrix f_* is the unique M s.t.

$$\hat{f} - M \text{ is bounded on } \mathbb{R}^N$$

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Pf of Coro:

1. ✓

2. $p \circ \hat{f}(x+z) = p \circ \hat{f}(x)$

$$\begin{tikzcd} \mathbb{R}^N \arrow[r, "\tilde{f}"] \arrow[d, "p"'] \arrow[dr, dashed] & \mathbb{R}^N \arrow[d, "p"] \\ \mathbb{T}^N \arrow[r, "f"'] & \mathbb{T}^N \end{tikzcd}$$

Hence $\hat{f}(x+z) - \hat{f}(x) \in \mathbb{Z}^N$

it is also continuous. so. $\underset{\substack{\parallel \\ f_*(z)}}{\hat{f}(x+z) - \hat{f}(x)}$ is constant on $\mathbb{R}^N$

$$\begin{aligned} f_*(z+z') &= \hat{f}(x+z+z') - \hat{f}(x) \\ &= \hat{f}(x+z+z') - \hat{f}(x+z) + \hat{f}(x+z) - \hat{f}(x) \\ &= f_*(z') + f_*(z). \end{aligned}$$

Hence f_* is a linear map on $\mathbb{Z}^N$, given by integer matrix

3. $f_* \Rightarrow \hat{f} - f_*$ bounded $\quad \hat{f}(x+z) - f_*(x+z) = \hat{f}(x) - f_*(x)$

uniqueness. any $M \neq 0$. M not bounded on $\mathbb{R}^N$

Def. distance on torus. $\mathbb{T}^N$

$$d(\theta, \theta') = \inf_{p(x)=\theta,\, p(x')=\theta'} \|x, x'\| , \quad \|\cdot\| \text{ euclidean norm on } \mathbb{R}^N$$

lem 7.10 f, g two maps on $\mathbb{T}^N$. if $\forall \theta$, $d(f(\theta), g(\theta)) < 1/2$

then $f_* = g_*$

Pf: $\hat{f}(x) - \tilde{g}(x)$ is contained in $\bigcup_{z \in \mathbb{Z}} B(z, \frac{1}{2})$, a disjoint union

By continuity. $\hat{f} - \tilde{g}$ stay in the same ball., uniformly bounded.

Hence by Coro 7.9 3). $f_* = g_*$

<u>Thm</u>. Let M be an invertible integer hyperbolic matrix. f_M torus automorphism. Then $\exists$ a neighborhood U of f_M on $C^1(\mathbb{T}^N, \mathbb{T}^N)$ s.t. $\forall f \in U$ top conjugate ~~to~~ f_M ($\exists h$ ~~top~~ homeo. s.t $f_M \circ h = h \circ f$).

<u>Pf</u>. ~~Take~~

By lem 7.10. If $\|f - f_M\|_{L^\infty(\mathbb{T}^N)} < 1/2$

then $f_* = M$

By Cor 7.9 iii) ~~$\Rightarrow$~~ $\hat{f} - M$ is bounded on $\mathbb{R}^N$

where $M = \hat{f}_M$

Want to apply <u>Prop 7.4</u> need to verify Lipschitz property of $\hat{f} - M$.

$$\mathrm{Lip}(\hat{f} - M) \le \sup_{\mathbb{R}^N} (D\hat{f} - M) = \sup_{\mathbb{T}^N} (Df - Df_M)$$

By shrinking U. $Df - Df_M$ is small. we can apply 7.4 to obtain $H = id + h$, s.t.

$$\cancel{H \circ M = \hat{f} \circ H} \quad M \circ H = H \circ \hat{f}$$

Need to verify H descends to $\mathbb{T}^N$.

<u>Lem</u>: $\forall x \in \mathbb{R}^N$, $H(x)$ is the unique pt $y \in \mathbb{R}^N$ s.t

$$\sup_{n \in \mathbb{Z}} \| M^n y - \hat{f}^n(x) \| < \infty$$

<u>Pf</u> 2f $y = H(x)$ ✓

For any $y' \neq y$. $\sup_{n \in \mathbb{Z}} \| M^n (y - y') \| = \infty$

due to M hyperbolic

$\square$

Need to verify $\forall x \in \mathbb{R}^N$, $z \in \mathbb{Z}^N$. $H(x+z) - H(x) \in \mathbb{Z}^n$

$H(x+z)$ is determined by

$$\sup_{n \in \mathbb{Z}} \| M^n y - \hat{f}^n (x+z) \| < \infty$$

Now
$$M^n (H(x)+z) - \hat{f}^n (x+z)$$
$$= \cancel{\hat{f}^n(x)} \, M^n H(x) + M^n z - \hat{f}^n (x+z)$$
$$= M^n H(x) - \hat{f}^n (x). \quad \Rightarrow \quad H(x+z) = H(x) + z.$$

By this way, we get h. , h^{-1} by the same way. $\square$