- Exponential mixing of circle expansion ①

Let $f_M : \mathbb{T} \to \mathbb{T}$. $x \mapsto Mx$ $|M| \geq 2$

φ, ψ. C^1 function on $\mathbb{T}$, then

$$\int \varphi \circ f_M^n \psi - \int \varphi \int \psi = O(|\varphi|_{C^1} |\psi|_{C^1} M^{-n}).$$

Pf. Use Fourier expansion, $e_k(x) = e^{2\pi i x k}$. $k \in \mathbb{Z}$

$$\varphi = \sum \varphi_k e_k, \quad \psi = \sum \psi_\ell e_\ell.$$

Due to $\varphi, \psi \in C^1$

$$\varphi_k = \int \varphi(x) e^{-2\pi i k x}(x) = \frac{1}{-2\pi i k} \int \varphi(x) \frac{d e^{-2\pi i k x}}{dx}$$

$$= \frac{1}{2\pi i k} \int \varphi'(x) e^{-2\pi i k x}.$$

$$|\varphi_k| \leq \frac{1}{k} |\varphi'|_{L^1} \leq \frac{1}{k} |\varphi|_{C^1}.$$

Therefore

$$\int \varphi \circ f_M^n \psi - \int \varphi \int \psi = \int \sum_k \varphi_k e_k \circ f^n \sum_\ell \psi_\ell e_\ell - \varphi_0 \psi_0$$

$$= \sum_{\substack{k, \ell \\ (k,\ell) \neq 0}} \varphi_k \psi_\ell \int e_k \circ f^n e_\ell$$

$$= \sum_{\substack{k, \ell \\ (k,\ell) \neq 0}} \varphi_k \psi_\ell \, \delta_{M^n k = \ell}. = \sum_{k \neq 0} \varphi_k \psi_{M^n k}.$$

$$|\varphi_k \psi_{M^n k}| \leq |\varphi|_{C^1} |\psi|_{C^1} / M^n k^2.$$

The error $\leq \sum_{|k| \neq 0} |\varphi|_{C^1} |\psi|_{C^1} / M^n k^2 \leq M^{-n} |\varphi|_{C^1} |\psi|_{C^1} \cdot C.$

Ques: What about higher dimensional case?
Especially, when there are eigenvalue in the unit circle (Partially hyperbolic).

~~Gauss map: For $x \in [0,1] \setminus \mathbb{Q}$. Let~~

$$\cancel{f(x) = \frac{1}{x} - \left[\frac{1}{x}\right]}$$

- Dynamical proof of mixing for cat map $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

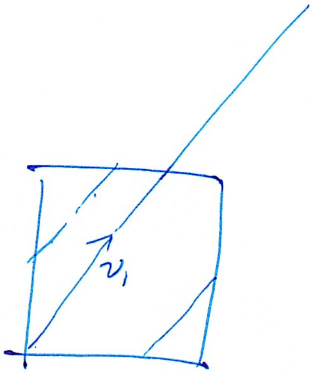
Let
$$v_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

then $M v_i = \lambda_i v_i$. $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$.

Therefore any $v \in \mathbb{R}^2$.

$$M^n v = M^n(\alpha v_1 + \beta v_2) = \alpha \lambda_1^n v_1 + \beta \lambda_2^n v_2.$$

So the map f_M preserves the foliation given by v_i. (expanding / contracting foliations)



Let $h_1^s(x) = x + s v_1$, then

$$(\star) \quad f_M^n \circ h_1^s(x) = f_M^n(x) + s\lambda_1^n v_1 = h_1^{s\lambda_1^n} \circ f_M^n(x)$$

Take φ, ψ C^1 functions on $\mathbb{T}^2$, then due to h_1^s preserving Leb.

$$I_n = \int \varphi \, \psi \circ f_M^n = \frac{1}{t} \int_0^t \varphi(h^s x) \, \psi \circ f_M^n(h_1^s x) \, ds \, d\mathrm{Leb}(x)$$

$$\approx \int \varphi(x) \left(\frac{1}{t} \int_0^t \psi \circ f_M^n \circ h_1^s(x) \, ds \right) d\mathrm{Leb}(x) + |t| \cdot |\varphi|_{C^1} |\psi|_\infty$$

Now
$$\frac{1}{t} \int_0^t \psi \circ f_M^n \circ h_1^s(x) \, ds = \frac{1}{t} \int_0^t \psi \, h_1^{\lambda_1^n s} f_M^n(x) \, ds$$

Hence due to f_M preserving Leb

$$\int \varphi(x) \frac{1}{t}\int_0^t \psi\, h_1^{\lambda_1^n s} f_M^n(x)\, ds \quad d\,Leb(x)$$

$$= \int \varphi(f_M^{-n}(x)).\ \underline{\frac{1}{t}\int_0^t \psi\, h_1^{\lambda_1^n s}(x)\, ds} \quad d\,Leb(x)$$

By ~~the~~ unique ergodic of h_1^1 $\Rightarrow$ flow unique ergodic

$$\frac{1}{t}\int_0^t \psi\, h_1^{\lambda_1^n s}(x)\, ds = \cancel{\frac{b}{t\lambda_1}}\ \frac{1}{t\lambda_1^n}\int_0^{t\lambda_1^n} \psi(h_1^s(x))\, ds$$

$$\longrightarrow \int \psi\, d\,Leb$$

Therefore

$$I_n \xrightarrow{n\to\infty} \int \varphi \int \psi .$$

<u>Rk</u>: Dynamical proof, use property of flow h_1^s and relation $(*)$

Similar idea also appears in unipotent flow and geodesic flow

- <u>Gauss Map</u>:

For $x \in (0,1]$ let $a_1(x) = [\frac{1}{x}]$, Gauss map $f(x) = \frac{1}{x} - [\frac{1}{x}]$

$$a_n(x) = a_1(f^{n-1}(x))$$

By this way, we obtain continued fraction expansion of x

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots \cfrac{1}{a_n + f^n(x)}}} = \frac{p_n + p_{n-1} x_n}{q_n + q_{n-1} x_n}.$$

with $x_n = f^n(x)$

Due to $x_n = \dfrac{1}{a_{n+1} + x_{n+1}}$

$$\frac{p_{n+1} + p_n x_{n+1}}{q_{n+1} + q_n x_{n+1}} = \frac{p_n(a_{n+1} + x_{n+1}) + p_{n-1}}{q_n(a_{n+1} + x_{n+1}) + q_{n-1}} = \frac{p_{n-1} + p_n a_{n+1} + p_n x_{n+1}}{q_{n-1} + q_n a_{n+1} + q_n x_{n+1}}$$

(*)
$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} \cancel{(} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} p_1 & p_0 \\ q_1 & q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}.$$

Lem: i) Continued fraction is finite iff $x \in \mathbb{Q}$

ii) the map $x \longmapsto (a_1(x), a_2(x), \ldots)$ is a bijection from $[0,1] \setminus \mathbb{Q} \to (\mathbb{N} - \{0\})^{\mathbb{N} - \{0\}}$

i) ✓

Pf: ii) For $a = (a_1, \ldots, a_n)$, let $\Delta_a = \{x \mid a_1(x), \ldots, a_n(x) = a\}$

Then $f^n(\Delta_n) = (0,1]$. and the inverse is given by

$\psi_a(x) = f^{-n}|_{\Delta_n}(x) = \dfrac{p_n + p_{n-1}(x)}{q_n + q_{n-1}(x)}$ (p, q given by (*))

Hence $\Delta_n = \left(\dfrac{p_n}{q_n}, \dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}\right)$ or $\left(\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}, \dfrac{p_n}{q_n}\right)$.

$$h'(x) = \frac{p_{n-1} q_n - p_n q_{n-1}}{(q_n + q_{n-1} x)^2} = \frac{(-1)^{n-1}}{(q_n + q_{n-1} x)^2}.$$

$$\text{length}(\Delta_n) = \left| \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n(q_n + q_{n-1})} \to 0$$

as $q_n \to \infty$. □

Ergodicity of Gauss map

Prop. The measure $\mu = \frac{1}{\ln 2} \frac{1}{1+x}\, dx$ is ergodic for $\underbrace{f}_{\text{Gauss map}}$.

Coro: For a.e. $x \in (0,1]$, for $\ell \in \mathbb{N} - \{0\}$

$$\frac{1}{n} \#\{1 \le k \le n \mid a_k(x) = \ell\} \xrightarrow{n \to \infty} \ln_2\left(1 + \frac{1}{\ell+1}\right)$$

Pf. Let $A_\ell = \{x \mid a_1(x) = \ell\}$ then $\mu(A_\ell) = \frac{1}{\ln 2}\left(\ln\left(1+\frac{1}{\ell}\right) - \ln\left(1+\frac{1}{\ell+1}\right)\right)$.

$$= \frac{1}{\ln 2} \ln\left(1 + \frac{1}{\ell(\ell+2)}\right)$$

By Birkhoff ergodic thm.

$$\frac{1}{n} \#\{1 \le k \le n \mid a_k(x) = \ell\} = \frac{1}{n} \#\{1 \le k \le n \mid f^{k-1}(x) \in A_\ell\}$$

$$\longrightarrow \mu(A_\ell).$$

$\square$

Step 1. μ is f-invariant.

For any $\varphi \in L^1$ we have

$$\int \varphi \circ f \, d\mu = \int_{(0,1]} \varphi(f(x)) \frac{1}{\ln 2} \frac{1}{1+x}\, dx$$

$$= \sum_n \int_{[\frac{1}{n+1}, \frac{1}{n}]} \varphi\left(\frac{1}{x} - n\right) \frac{1}{\ln 2} \frac{1}{1+x}\, dx$$

change of variable

$$= \sum_n \int_{[0,1]} \varphi(y) \frac{1}{\ln 2} \frac{1}{1 + 1/(y+n)} \frac{1}{(y+n)^2}\, dy$$

$$= \sum_n \int_{[0,1]} \varphi(y) \frac{1}{\ln 2} \left(\frac{1}{y+n} - \frac{1}{y+n+1}\right) dy$$

$$= \int_{[0,1]} \varphi(y) \frac{1}{\ln 2} \frac{1}{1+y}\, dy.$$

$\square$

Step 2: Renyi condition

~~For any~~ $\exists\, c, c' > 0$, for any Borel set A and $a = (a_1, \dots, a_n) \in \mathbb{N}^n$. we have

$$c\,\mu(\Delta_a)\,\mu(A) \le \mu(\Delta_a \cap f^{-n}(A)) \le c'\,\mu(\Delta_a)\,\mu(A),$$

Pf: μ and Leb differ by a bounded density. suffices to consider Leb.
Notice the map

$$f^{-n} : \Delta_a \longrightarrow (0, 1]$$

is a diffeo and

$$\psi_a(x) = f^{-n}|_{\Delta_a}(x) = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x} \qquad \text{inverse branch}$$

$$\left|\left(f^{-n}|_{\Delta_a}\right)'(x)\right| = \left|\frac{(-1)^{n-1}}{(q_n + q_{n-1}x)^2}\right| \in \left(\frac{1}{(q_n+q_{n-1})^2}, \frac{1}{q_n^2}\right)$$

$$\text{Leb}(\Delta_a) = \left|\frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right| = \frac{1}{q_n(q_n + q_{n-1})}$$

Due to $q_n > q_{n-1}$, $\exists\, c, c' > 0$.

$$c\,(f^{-n}|_{\Delta_a}(x))' \le \text{Leb}(\Delta_a) \le c'\,(f^{-n}|_{\Delta_a}(x))'$$

Hence

$$\text{Leb}(\Delta_a \cap f^{-n}(A)) = \text{Leb}(\psi_a(A)) = \int_A |\psi_a'(x)|\,dx$$

$$\in \text{Leb}(A)\left(\min \psi_a, \max \psi_a\right)$$

$$\in \text{Leb}(A)\,\text{Leb}(\Delta_a)\,(c, c')$$

$\square$

Let Y be a f-inv Borel set

Step 3. Let $\nu(A) = \mu(A \cap Y) - \frac{c}{2}\mu(A)\mu(Y)$

with c from (Renyi Condition)

For $A = \Delta_a$ we have

$$\nu(\Delta_a) = \mu(\Delta_a \cap Y) - \frac{c}{2}\mu(\Delta_a)\mu(Y)$$
$$= \mu(\Delta_a \cap f^{-n}(Y)) - \frac{c}{2}\mu(\Delta_a)\mu(Y) \geq 0$$
$$\geq \frac{c}{2}\mu(\Delta_a)\mu(Y)$$

(Δ_a) a Bool semi-algebra generates Borel σ-algebra

$\forall A$ Borel $\forall \varepsilon > 0$. $\exists A'$ finite union of Δ_a

$$\mu(A \Delta A') < \varepsilon$$

So $\forall A$ Borel

$$\nu(A) \geq 0.$$

Take $A = Y^c$. $\nu(Y^c) = -\frac{c}{2}\mu(Y^c)\mu(Y) \geq 0$

Hence $\mu(Y)$ or $\mu(Y^c) = 0$ $\square$

- Thm Gauss map is mixing.

$\exists c, c' > 0$

Lem: For any Borel sets A, B

$$c'\mu(A)\mu(B) \leq \liminf_n \mu(A \cap f^{-n}B) \leq \limsup_n \mu(A \cap f^{-n}(B)) \leq c\,\mu(A)\mu(B)$$

Pf: For $\Delta = \Delta_a$, $\forall m > n$. we can decompose

$$\Delta_a = \bigsqcup_{\text{length}(b) = m} \Delta_b ,$$

By Renyi condition $c'\mu(A)\mu(B) \leq \mu(\Delta_a \cap f^{-m}(B)) \leq c\,\mu(A)\mu(B)$

Then by approximation of Borel sets $\square$ (8)

Pf of mixing: Let $\nu = \mu \times \mu$. and $g = f \times f$.

Let $\nu_n = (id \times f^n)_* \mu$

Only need to prove for any weak limit ν_∞ of $\{\nu_n\}_n$ we have $\nu_\infty = \nu$.

Step 1: ν is g - ergodic

Pf: For any cylinder set $P = A \times B$, $Q = A' \times B'$.

we have $\liminf \nu(P \times g^{-n} Q)$

$= \liminf \mu(A \times f^{-n} A') \mu(B \times f^{-n} B')$

$\geq \cancel{(c, c')} c \; \nu(P) \nu(Q)$.

Use cylinder sets to approximate Borel sets

we have $\liminf \nu(P \times g^{-n} Q) \cancel{\in (c, c')} \nu(P) \nu(Q)$ $\geq c$

For any g - inv set Y.

$0 = \liminf \nu(Y \times g^{-n} Y) \geq c \; \nu(Y) \nu(Y')$. $\square$

Step 2: $\nu_\infty(A \times B) \leq \limsup \nu_n(A \times B)$

$= \limsup \mu(A \times f^{-n} B) \leq C' \mu(A) \mu(B)$

$= C' \nu(A \times B)$

$\Rightarrow \forall P$ Borel $\nu_\infty(P) \leq \nu(P)$

Hence ν_∞ is absolute continuous w.r.t ν.

$$d\nu_\infty(x) = \cancel{d\nu(x)}\,\rho(x)\, d\nu(x)$$

for $\rho \in L^1(\nu)$. by Radon-Nikodym thm.

But $\rho\,\nu_\infty$ is g-inv. and ν is g-erg.

Hence $\rho = cst$ a.e. So $\nu_\infty = \nu$. $\square$

Growth rate of continued fraction

Thm (Borel-Bernstein). Let $(k_n)_{n\geq 1}$ be an increasing sequence of positive numbers. $E = \{x \in X \mid \forall n_0 \geq 1.\ \exists n \geq n_0\ a_n(x) \geq k_n\}$

i) If $\sum \frac{1}{k_n} < \infty$, then E is of zero measure

ii) If $\sum \frac{1}{k_n} = \infty$, then E is of measure 1

This tells us a.e. $x \in [0,1]$. $a_n(x) = O(n^{1+\varepsilon})$

Zero-One law. but not $O(n \log n)$

Pf: i) Similar to Borel-Cantelli:

Let $A_n = \{x \mid a_n(x) \geq k_n\}$. then

$$E = \bigcap_{j\geq 1} \Big(\bigcup_{n\geq j} A_n \Big)$$

$$\mu(A_n) = \mu\{x \mid a_n(x) \geq k_n\} \overset{\text{invariance}}{=} \mu\{x \mid a_1(x) \geq k_n\} \asymp 1/k_n$$

Hence by $\sum 1/k_n < \infty$

$$\sum_{n \geq j} \mu(A_n) \leq \sum_{n \geq j} 1/k_n \xrightarrow{\text{as } j \to \infty} 0 .$$

Therefore $\mu(E) = 0$

(Borel-Cantelli, need independence)

ii) For any $n_0 \in \mathbb{N}$

Let $C_n = \bigcup_{j=n_0}^{n} A_j$, $C_n \subset C_{n+1}$

Then

$$\mu(\cancel{C_n} \, C_{n+1}^c) = \mu(C_n^c) - \mu(C_n^c \cap A_{n+1})$$

From the construction, we know

C_n^c can be represented as a countable union of Δ_a with $a = (a_1, \dots, a_n)$.

Hence we can apply lem of Renyi to obtain

$$\mu(C_n^c \cap A_{n+1}) \geq c \, \mu(C_n^c) \, \mu(A_{n+1})$$

Therefore

$$\mu(C_{n+1}^c) \leq \mu(C_n^c)(1 - c\,\mu(A_{n+1}))$$

$$\leq \mu(C_n^c)\left(1 - \frac{c}{k_{n+1}}\right)$$

$$\mu(C_n^c) \leq \mu(C_{n_0}^c)\left(1 - \frac{c}{k_{n_0}}\right)\left(1 - \frac{c}{k_{n_0+1}}\right) \cdots \left(1 - \frac{c}{k_n}\right)$$

Due to $\sum \frac{1}{k_n} = \infty$, $\prod_{j \geq n} \left(1 - \frac{c}{k_j}\right) = 0$

$$\mu(C_n^c) \xrightarrow{\text{as } n \to \infty} 0$$