Mixing

Recall: $(X, \mathcal{B}, \mu) \curvearrowleft f$. measure preserving

Ergodic: $\iff \forall\, A, B \in \mathcal{B}$ $\lim \frac{1}{n} \sum_{0\le m\le n-1} \mu(f^{-m}(A)\cap B) = \mu(A)\mu(B)$

Def: $\Uparrow$

Weakly mixing — $\lim \frac{1}{n} \sum_{0\le m\le n-1} |\mu(f^{-m}(A)\cap B) - \mu(A)\mu(B)| = 0$

$\Uparrow$

Mixing: — $\lim \mu(f^{-m}(A)\cap B) = \mu(A)\mu(B)$

asymptotic independence

Thm: 1) Weakly mixing

$\iff$ 2) $f\times f$ on $(X\times X, \mu\times\mu)$ is ergodic

$\iff$ 3) U_f on $L^2(X,\mu)$ has no eigenfunction other than 1.

$U_f \varphi = \varphi \circ f$. a unitary operator

Pf: 1) $\Rightarrow$ 2). Only consider ~~set~~ product sets $A\times B$ ⊕. (Borel semi-alg)

$A_1\times A_2$ and $B_1\times B_2$. compute

$$\frac{1}{n}\sum_{0\le m\le n-1} \mu\times\mu\,(f^{-m}(A_1\times A_2)\cap B_1\times B_2)$$

$$= \frac{1}{n}\sum_{0\le m\le n-1} \mu(f^{-m}(A_1)\cap B_1)\,\mu(f^{-m}(A_2)\cap B_2)$$

$$= \frac{1}{n}\sum_{0\le m\le n-1} \big(\mu(f^{-m}(A_1)\cap B_1) - \mu(A_1)\mu(B_1)\big)\,\mu(f^{-m}(A_2)\cap B_2)$$

$$+ \mu(A_1)\mu(B_1)\,\frac{1}{n}\sum_{0\le m\le n-1} \mu(f^{-m}(A_2)\cap B_2))$$

$$\xrightarrow{} \underset{\text{weakly mixing}}{0} + \underset{\text{ergodic}}{\mu(A_1)\mu(B_1)\mu(A_2)\mu(B_2)}$$

2) $\Rightarrow$ 1) ~~Take~~ 2) $\Rightarrow$ f is ergodic. 2

Then take $A\times A$, $B\times B$.

$$\mu(A)^2\mu(B)^2 = \lim \frac{1}{n} \sum_{0\le m\le n-1} \mu(f^{-n}(A\times A) \cap (B\times B))$$

$$= \lim \frac{1}{n} \sum_{0\le m\le n-1} \mu(f^{-n}(A)\cap B)^2$$

$$= \lim \frac{1}{n} \sum_{0\le m\le n-1} \left(\mu(f^{-n}(A)\cap B) - \mu(A)\mu(B)\right)^2 \mp \mu(A)^2\mu(B)^2$$

$$+ \quad 2\left(\lim \frac{1}{n}\sum \mu(f^{-n}(A)\cap B)\right).\mu(A)\mu(B)$$

$$\Rightarrow \lim \frac{1}{n}\sum \left(\mu(f^{-m}(A)\cap B) - \mu(A)\mu(B)\right)^2 = 0$$

__lem.__ Let a_n be a bounded sequencen in $\mathbb{R}$. then

1) $\frac{1}{n}\sum |a_n| = 0$

2) $\frac{1}{n}\sum |a_n|^2 = 0$

3) $\exists\, J\subset \mathbb{N}$ zero density s.t. $\lim\limits_{n\in\mathbb{N}/J} a_n = 0$

1) => 3) If U_f has other eigenfunction. $L^2(X,\mu,\mathbb{C})$

$U_f\varphi = \lambda\varphi$, since $\langle U_f\varphi, U_f\varphi\rangle = \langle\varphi,\varphi\rangle$

$\Leftrightarrow \int |\varphi\circ f|^2 d\mu = \int |\varphi|^2 d\mu$

Unitary, ~~so~~ $|\lambda| = 1$. and $\lambda\ne 1$, due to ergodicity

$\lambda\int\varphi = \int \varphi\circ f\, d\mu = \int\varphi \quad \Rightarrow \int\varphi = 0$

$$\lim_{n\to\infty} \frac{1}{n}\left|\int \varphi\circ f^n . \bar{\varphi}\right| = \int|\varphi|^2 \ne 0$$

3) $\Rightarrow$ 1) spectral theory

Prop. Mixing is equivalent to.

$\forall\ \psi, \varphi \in L^2(X, \mu)$

$$\lim \int \psi \circ f^n\ \varphi\ d\mu = \int \psi \int \varphi .$$

• Moreover, sufficient to verify ψ, φ in a dense subspace

Pf: $\Leftarrow$ Take $\psi = 1_A$, $\varphi = 1_B$. then

$$\int \psi \circ f^n \varphi\ d\mu = \int 1_A (f^n(x))\ 1_B(x)\ d\mu$$

$$= \mu(f^{-n}(A) \cap B) \quad \checkmark$$

$\Rightarrow$ ψ, φ sum of step functions

$\psi = \sum a_i 1_{A_i}$ $\quad \varphi = \sum b_j 1_{B_j}$. $\quad A_i$ disjoint, B_j disjoint

$$\int \psi \circ f^n \varphi\ d\mu = \sum_{i,j} a_i b_j \int \mu (f^{-n}(A_i) \cap B_j).$$

$$\xrightarrow{n \to \infty} \sum_{i,j} a_i b_j\ \mu(A_i)\ \mu(B_j) = \int \varphi \int \psi .$$

Density of step functions. Stone-Weierstrass.

$$\int \psi_k \circ f^n \varphi_k - \int \psi \circ f^n \varphi$$

$$\leq \int (\psi_k - \psi) \circ f^n \varphi_k - \int \psi \circ f^n (\varphi - \varphi_k)$$

Cauchy-Schwartz

$$\leq \left(\int |(\psi_k - \psi) \circ f^n|^2 \int |\varphi_k|^2\right)^{1/2} + \left(\int |\psi \circ f^n|^2 \int |\varphi - \varphi_k|^2\right)^{1/2} \to 0$$

Example: Circle rotations

• Torus transformation $\qquad M$ integer matrix.

M:
$$\begin{tikzcd} \mathbb{R}^n \arrow[r] \arrow[d, "p"'] & \mathbb{R}^n \arrow[d, "p"] \\ \mathbb{T}^n \arrow[r, "f_M"] & \mathbb{T}^n \end{tikzcd}$$

lem. If $\det M \neq \phi$, then

i) f_M is surjective

ii) f_M preserves the Lebesgue measure $d\theta$

iii) $\forall \theta$, $\#(f_M^{-1}\theta) = |\det M|$.

pf. i) $M: \mathbb{R}^n \to \mathbb{R}^n$ surjective.

ii) $R_{M\alpha} \circ f_M = f_M \circ R_\alpha$.. $\alpha = (\alpha_1, \cdots, \alpha_n)$ rational ind.

Then $R_{M\alpha}$ preserves the measure $(f_M)_* d\theta$.

But $R_{M\alpha}$ unique ergodic $\Rightarrow$ $(f_M)_* d\theta = d\theta$

iii) $\forall \theta$ $\quad p^{-1}(\theta) = \mathbb{Z}^n + \theta$

$$M^{-1}(p^{-1}\theta) = M^{-1}\mathbb{Z}^n + M^{-1}\theta$$

Hence the preimages given by $M^{-1}\theta + M^{-1}\mathbb{Z}^n/\mathbb{Z}^n$.

$M^{-1}\mathbb{Z}^n/\mathbb{Z}^n \simeq \mathbb{Z}^n/M\mathbb{Z}^n$. size $|\det M|$.

Smith normal form $\to$ $M = M_1 \operatorname{diag} M_2$. $\quad M_1, M_2 \in SL_n(\mathbb{Z})$

index theory ?

Prop. Let f_M given before, then TFAE

i). f_M is ergodic

ii) f_M is mixing

iii) no root of unity is an eigenvalue of M.

Pf: ii) $\Rightarrow$ i) ✓

i) $\Rightarrow$ iii) If not. $\exists\, i$. s.t. M^i has eigenvalue 1

then $M^i v = v$ has integer solution,

Let $\psi = e_{v^t} + e_{(Mv)^t} + \cdots + e_{(M^{i-1}v.)^t}$ $\quad e_v := e^{2\pi i(x, v^t)}$

$$e_v \circ f_M(x) = e^{2\pi i \langle f_M(x), v^t\rangle} = e^{2\pi i \langle x, (Mv)^t\rangle} = e_{(Mv)^t}$$

$\psi \circ f_M = \psi$. but $\psi \neq 0$ contradiction

iii) $\Rightarrow$ ii) By Stone-Weierstrass + linearity, sufficient for ψ, φ. both e_v, e_w.

then
$$\int \varphi \circ f_M^n \psi = \int e_v \circ f_M^n e_w$$
$$= \int e_{M^n v} e_w. = \int e_{M^n v + w}.$$

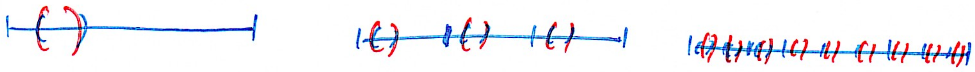
$v, Mv, M^2v, \cdots$, are different, so. $M^n v + w \neq 0$ for n large

$$\Rightarrow \lim \int \varphi \circ f_M^n \psi = 0$$

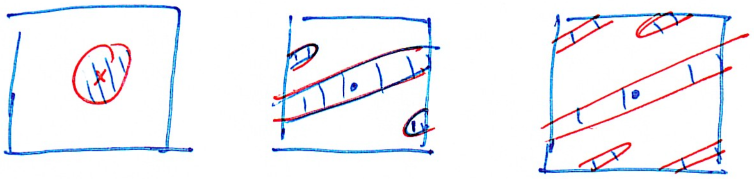
Geometric intuition.

$f(x) = 3x$ on $\mathbb{T}$.

A $\qquad$ $f^{-1}(A)$ $\qquad$ $f^{-2}(A)$



$f(x) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. on $\mathbb{T}^2$



B $\qquad$ $f_M^2(B)$ $\qquad$ $f_M^4(B)$

<u>example</u>: Symbolic dynamics. cylinder sets

$$\mu(f^{-n}(A) \cap B) = \mu(A)\mu(B) \quad \text{for } n \gg 1$$