## (一) Generic pts.



**Def.** $(X, \mathcal{B}, \mu)$. $f$. continuous. measure preserving.

$x$ generic, if $\forall\ \varphi \in C(X)$

$$S_n \varphi(x) \xrightarrow{n\to\infty} \int \varphi\, d\mu.$$

Riesz Rep $\Longleftrightarrow$

$$\frac{1}{n} \sum_{0\le k\le n-1} \delta_{f^k(x)} \longrightarrow \mu.$$

**Thm.** $X$ cpt metric space. $\mu$ is $f$-ergodic

Then $\mu$. a.e. $x$ is generic

**Pf:** (Stone - Weierstrass). $C(X)$, $\exists$ a countable subset.

$\{\varphi_k\}$. s.t. $\overline{\{\varphi_k\}} = C(X)$ (infinity norm)

$\forall\ \varphi_k$ By Birkhoff ergodic

$$\mu.\ \text{a.e.}\ x \quad S_n \varphi_k(x) \xrightarrow{n\to\infty} \int \varphi_k\, d\mu$$

Take countable intersection

$$\mu.\ \text{a.e}\ x. \quad \forall k,\quad S_n \varphi_k(x) \xrightarrow{n\to\infty} \int \varphi_k\, d\mu$$

The $\forall\ \varphi \in C(X)$. $\exists\ \varphi_{k_\ell} \xrightarrow{\|\ \|_\infty} \varphi$.

$$| S_n\varphi(x) - S_n \varphi_{k_\ell}(x) | \le |\varphi - \varphi_{k_\ell}|_\infty \xrightarrow{k_\ell\to\infty} 0$$

$$| S_n \varphi_{k_\ell}(x) - \int \varphi_{k_\ell} | \xrightarrow{n\to\infty} 0$$

$$| \int \varphi_{k_\ell}\, d\mu - \int \varphi\, d\mu | \le |\varphi - \varphi_{k_\ell}|_\infty \xrightarrow{k_\ell\to\infty} 0$$

$$\Longrightarrow \quad S_n \varphi(x) \xrightarrow{n\to\infty} \int \varphi\, d\mu \qquad \square$$

**Coro 1:** $\forall\ \mu$ f-ergodic. then $\forall\ \mu$. a.e. $x$

$$\frac{1}{n}\sum_{0\le k\le n-1}\delta_{f^k(x)} \longrightarrow \mu.$$

**Coro 2:** Birkhoff recurrence thm. $\forall X$ cpt metric space, $f$-continuous

(without axiom of choice)

**Pf.** $\exists\ \mu$. Borel measurable, $f$-inv.

$\mathcal{M}(X, f) = \{ f\text{-inv. Borel measurable measure} \} \neq \emptyset$.

Take an extremal point $\nu$. then $\nu$ f-ergodic,

Coro 1 $\Rightarrow$ $\nu$ a.e. $x$. $\frac{1}{n}\sum_{0\le k\le n-1}\delta_{f^k(x)} \to \nu$.

Take one such $x_\nu \in \operatorname{supp}\nu$.

$\forall$ open $V \ni x$, $\nu(V) > 0$. then

$$\Big(\frac{1}{n}\sum_{0\le k\le n-1}\delta_{f^k(x)}\Big)(V) = \frac{1}{n}\#\{k \le n-1 \mid f^k(x) \in V\}. \to \nu(V) > 0$$

$\Rightarrow\ \exists\ k>0,\ f^k(x) \in V.$ $\square$

## 三 Uniquely ergodic

(every pt is generic)

**Def.** $(X, \mathcal{B}, \mu)$. $f$-cont, $\mu$-ergodic.,

If for every $x \in X$. $\forall\ \varphi \in C(X)$.

$$S_n\varphi(x) \longrightarrow \int \varphi\, d\mu.$$

Then $f$ is <u>uniquely ergodic.</u>

**Thm.** (TFAE, the following are equivalent $(X, f)$ cpt metric

1. $\forall\ \varphi \in C(X)$. $S_n\varphi$. converges to a constant <u>uniformly</u>
2. $\forall\ \varphi \in C(X)$. $S_n\varphi$. ———— — — . at every pt

3. $\exists$ unique $f$- invariant measure $\mu$

4. $f$ uniquely ergodic

Pf. 1) $\Rightarrow$ 2). f-inv ergodic

2) $\Rightarrow$ 3.) If $\exists$ $\mu_1 \neq \mu_2$. $\exists$ $\varphi$. $C(X)$ s.t. $\int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2$

Birkhoff ergodic.

$$\mu_i \text{ a.e. } x \quad \oint S_n\varphi(x) \xrightarrow{n\to\infty} \int \varphi \, d\mu_i$$

$\exists$ $x_1, x_2$, s.t.

$$S_n\varphi(x_1) \longrightarrow \int \varphi \, d\mu_1$$
$$\neq$$
$$S_n\varphi(x_2) \longrightarrow \int \varphi \, d\mu_2 .$$

contradiction

3) $\Rightarrow$ 1) If not, $\exists$ $\varphi$, $\varepsilon > 0$. s.t. $\forall N$

$\exists$ $n > N$, $x_n \in X$.

$$\left| S_n\varphi(x_n) - cst \right| > \varepsilon$$

$$S_n\varphi(x_n) = \int \varphi \; d\Big(\frac{1}{n}\sum_{0\le k\le n-1} \delta_{f^k(x_n)}\Big).$$

$$\mu_n = \frac{1}{n}\sum_{0\le k\le n-1} \delta_{f^k(x_n)}.$$

By Banach-Alaoglu, $\mu_{n_j} \to \mu_\infty$, f-inv.

$$\left| \int \varphi \, d\mu_\infty - cst \right| = \lim_{n_j\to\infty} \left| \int \varphi \, d\mu_{n_j} - cst \right| \geq \varepsilon$$

By Birkhoff erg. $cst = \int \varphi \, d\mu$. , then $\mu_\infty \neq \mu$.

Contradiction. $\square$

1) $\Rightarrow$ 4) $\Rightarrow$ 3)

三 Examples: 

1). Circle rotation. ~~$\alpha \notin \mathbb{Q}$.~~ $\alpha \in \mathbb{R} - \mathbb{Q}$., $R_\alpha$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is uniquely ergodic

(Thm (2))

Pf. $\boxed{\text{Weyl créteria}}$: Let ~~Df~~ $\{x_n\}_{n \in \mathbb{N}}$., $x_n \in \mathbb{R}/\mathbb{Z}$.

$\{x_n\}_{n \in \mathbb{N}}$ equidistribute to Leb

i.e. $\mu_n = \frac{1}{n} \sum_{1 \le k \le n} \delta_{x_k} \xrightarrow{n \to \infty}$ Leb.

$\Leftrightarrow$ $\forall\, m \in \mathbb{Z}$, $m \neq 0$.

$$\frac{1}{n} \sum_{1 \le k \le n} e^{2\pi i m x_k} \xrightarrow{m \to \infty} 0$$

$$\int e^{2\pi i m \cdot}\, d\Big(\frac{1}{n} \sum_{1 \le k \le n} \delta_{x_k}\Big) = \int e^{2\pi i m \cdot}\, d\mu_n$$

Apply to $R_\alpha$: $x_n = x + n\alpha$.

$$\Big| \frac{1}{n} \sum_{1 \le k \le n} e^{2\pi i m (x + k\alpha)} \Big|$$

$$= \Big| \frac{1}{n} e^{2\pi i m x} \frac{1 - e^{2\pi i m n\alpha}}{1 - e^{2\pi i m \alpha}} \Big| \xrightarrow{n \to \infty} 0 \qquad \square$$

2). Torus rotation: $\mathbb{T}^n \circlearrowleft R_\alpha$.

$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

$R_\alpha : x \mapsto x + \alpha$

Thm: 1. $1, \alpha_1, \cdots, \alpha_n$ rational independent

2. $R_\alpha$ is minimal

3. $R_\alpha$ is uniquely ergodic

Pf. (Fourier analysis)

2) $\Rightarrow$ 1). If not. $\exists\ m_j \in \mathbb{Z}$. $m = (m_1, \cdots, m_n) \neq 0$.

s.t. $\langle m, \alpha \rangle \in \mathbb{Z}$.

Consider $X_m = \{ x \in \mathbb{T}^n \mid \langle m, x \rangle \in \mathbb{Z} \}$.

$X_m \neq \mathbb{T}^n$. $X_m \neq \phi$. $X_m$ is $R_\alpha$-inv. closed.

Contradiction $\square$

1) $\Rightarrow$ 3). $\forall\ \mu$ · $R_\alpha$-inv.

$$\hat{\mu}(m) := \int_{\mathbb{T}^n} e^{2\pi i \langle m, x \rangle} d\mu(x)$$

$$= \int_{\mathbb{T}^n} e^{2\pi i \langle m, x+\alpha \rangle} d\mu(x)$$

$$= e^{2\pi i \langle m, \alpha \rangle} \hat{\mu}(m)$$

By $1, \alpha_1, \cdots, \alpha_n$ rational independent $\Rightarrow$ $\forall\ m \neq 0$, $\langle m, \alpha \rangle \notin \mathbb{Z}$

$\Rightarrow$ $e^{2\pi i \langle m, \alpha \rangle} \neq 1$ for $m \neq 0$.

$\Rightarrow$ $\hat{\mu}(m) = 0$ for $m \neq 0$. $\Rightarrow \mu = \text{Leb}$

3) $\Rightarrow$ 2). $\forall\ x \in \mathbb{T}^n$. $\frac{1}{n} \sum_{0 \le k \le n-1} \delta_{R_\alpha^k x} \longrightarrow \text{Leb}$

Hence. $\overline{\{ R_\alpha^k x,\ k \ge 0 \}} = \mathbb{T}^n$.

3). Skew product and equidistribution of $\{\{P(n)\}\}$

$\mathbb{T}^n \longrightarrow \mathbb{T}^n$

$$f(x) = (x_1 + \alpha, x_2 + x_1, x_3 + x_2, \cdots, x_n + x_{n-1})$$

Thm: If $\alpha \in \mathbb{R} - \mathbb{Q}$, then $f$ uniquely ergodic

Rmk: Nice way to construct new example

<u>Coro</u>: Let $P$ be a polynomial with leading coefficient irrational. Then $(\{P(k)\})_k$ equidistribute ~~to~~ on $[0,1]$.

<u>Pf</u>. Suppose $\deg P = n$. Let $P_n(X) = P(X)$.

$$P_j(X) = P_{j+1}(X+1) - P_{j+1}(X) \qquad 1 \le j \le n-1.$$

Then $P_1(X) = n!\, a_0 X$, where $a_0$ Leading coefficient.

Then with $\alpha = n!\, a_0$

$$f(P_1(k), \cdots, P_n(k))$$
$$= (P_1(k)+\alpha,\ P_1(k)+P_2(k),\ \cdots,\ P_{n-1}(k)+P_n(k))$$
$$= (P_1(k+1),\ P_2(k+1),\ \cdots,\ P_n(k+1))$$

By <u>Thm</u>. the sequence $\{(P_1(k), \cdots, P_n(k)),\ k \in \mathbb{N}\}$ equidistribute on $\mathbb{T}^n$.

In particular $\{P(k)\}_{k\in\mathbb{N}} = \{P_n(k)\}_{k\in\mathbb{N}}$. equidistribute on $\mathbb{T}$.

$\square$

<u>Pf of thm</u>. $\hat{f}(x,s) = (f(x),\ s+\psi(x))$

<u>Lem</u> (Furstenberg): If

$$\begin{array}{c} \text{ergodic } \hat{f} \curvearrowright (X\times\mathbb{T},\ m\times Leb) \\ \downarrow \pi \\ \text{unique ergodic } f \curvearrowright (X, m) \end{array}$$

$\pi_*(m\times Leb) = Leb$

~~$f\circ\pi = \pi\circ\hat{f}$~~.

Then ~~$m\times Leb$ is~~ $\hat{f}$ also unique ergodic

<u>Pf</u>: By <u>def</u>. we can write ~~$\hat{f}(x,s) = (f(x),\ \psi(x,s))$~~

Let $g_t(x,s) = (x, s+t)$, then $g_t$ commutes with $\hat{f}$

For a $\nu_0 = \nu$ $\hat{f}$-invariant measure,

let $\nu_t = (g_t)_* \nu_0$, then also $\hat{f}$-invariant

Claim: $\int_{\mathbb{T}} \nu_t \, dt = m \times \mathrm{Leb}$

Then by $m \times \mathrm{Leb}$ ergodic $\Rightarrow$ $\nu_t = m \times \mathrm{Leb}$, a.e. $t$

Proof of claim: $\forall \varphi \in C(\hat{X})$, then

$$\int \varphi \, d\left(\int_{\mathbb{T}} \nu_t dt\right) = \int \varphi(x, s+t) \, d\nu_0(x,s) \, dt$$

$$= \int \varphi(x, s+t) \, dt \, d\nu_0(x,s)$$

$$= \int \varphi(x, t) dt \, d\nu_0(x,s) = \int \varphi \, dt \, dm$$

Due to $f$-unique ergodic, $\pi_* \nu_0$ $f$-inv, so, $\pi_* \nu_0 = m$

$\square$

Back to thm: Induction on dim $n$ of $\mathbb{T}^n$,

By lem, it suffices to prove. Leb is $f$-ergodic

Take $\varphi$ $f$-inv. $L^2$. then

$$\hat{\varphi}(m) = \int e^{2\pi i \langle m, x\rangle} \varphi(x) \, dx$$

$$= \int e^{2\pi i \langle m, f(x)\rangle} \varphi(x) \, dx$$

$$= e^{2\pi i m_1 \alpha} \hat{\varphi}(m_1 + m_2, m_2 + m_3, \dots, m_{n-1} + m_n, m_n).$$

Then $|\hat{\varphi}(m)| = |\hat{\varphi}(m_1 + m_2, \dots, m_{n-1} + m_n, m_n)|$

If $m_n \neq 0$. $\Rightarrow$ & $|\hat{\varphi}(m)| \neq 0$, infinite term with equal non-zero absolute value. not $\ell^2(\mathbb{Z}^n)$

So $\hat{\varphi}(m) = 0$, continue argument with $m_{n-1} \dots$

$\Rightarrow$ $\hat{\varphi}(m) = 0$ if $m \neq 0$ $\square$

VII. Example of minimal, not uniquely ergodic

( Furstenberg 67) $\psi$ analytic.

$$\mathbb{T}^2 \to \mathbb{T}^2$$

$$f(x, y) = (x+\alpha,\ y+\psi(x)).$$

Key pt of proof. $f_0(x, y) = (x+\alpha,\ y).$

$f(x, y)$ ~~not~~ is measurable conjugate to $f_0$,

$f$ not continuous conjugate to $f_0$,

$$\Longleftrightarrow \quad \psi(x) = \varphi(x+\alpha) - \varphi(x).$$

has measurable solution $\varphi$.

no continuous solution $\varphi$.