Ergodicity

(Boltzmann.
ergon " energy, work"
hodos " way, path")

(一)

$(X, \mathcal{B}, \mu) \circlearrowleft f$ μ probability measure

Def. f is ergodic if $\forall A \in \mathcal{B}$, $f^{-1}(A) = A$.

$$\Rightarrow \mu(A) = 1 \text{ or } 0.$$

(Stronger version. $\mu(f^{-1}A \,\triangle\, A) = 0 \Rightarrow \mu(A) = 1$ or 1)

$\varphi : X \to \mathbb{C}$ measurable. – φ is f-invariant

if $\varphi \circ f = \varphi$.

– φ is a.e f-invariant if $\varphi \circ f = \varphi$ a.e. μ

Prop. TFAE (The following are equivalent)

1. f ergodic
2. $\forall \varphi$ measurable a.e f-invariant $\Rightarrow$ φ a.e. const function.
3. $\forall \varphi \in L^1(X,\mu)$. a.e. f-inv $\Rightarrow \varphi =$ constant a.e.
4. $\forall \varphi \in L^2(X,\mu)$. a.e f-inv $\Rightarrow \varphi =$ constant a.e.

Pf. $L^2 \subset L^1$, $4) \Rightarrow 3)$

$3) \Rightarrow 2)$, $\varphi \circ f = \varphi$ a.e. $\varphi_N = \min\{\varphi, N\} \in L^1$.

$\overset{3)}{\Rightarrow} \varphi_N \circ f = \varphi$ a.e. $\Rightarrow \varphi_N = cst$ a.e.

$2) \Rightarrow 1)$. $f^{-1}(A) = A$, $\varphi = 1_A \Rightarrow \varphi \circ f = \varphi \overset{2)}{\Rightarrow} \varphi = cst.$ a.e.

$1) \Rightarrow 4)$ $A = \{x \mid a < \varphi(x) < b\}$

$\varphi \circ f = \varphi$ a.e. μ. $\Rightarrow \mu(f^{-1}(A) \triangle A) = 0$ (2)

$B = \bigcup_{k \geq 0} f^{-k}(A)$. $f^{-1}(B) \subset B$. , $\mu(f^{-1}(B)) = \mu(B) = \mu(A)$

$\mu(B) = \mu(A)$.

$C = \bigcap_{\ell \geq 0} f^{-\ell}(B)$. $f^{-1}(C) = C$, $\mu(C) = \mu(B)$ $\square$

Prop: X compact metric space, μ probability, $\operatorname{supp} \mu = X$, f preserves μ, continue. The μ. a.e. x

$$\overline{\{f^n(x), n \in \mathbb{N}\}} = X$$

Pf. X compact metric space $\Rightarrow \exists \mathcal{C}$ countable family of open sets. top base. $\forall U_k \in \mathcal{C}$.

$$A_k = \bigcup_{n \geq 0} f^{-n}(U_k) \quad (x, \exists n \text{ s.t } f^n(x) \in U_k)$$

$f^{-1}(A_k) \subset A_k \xRightarrow{\boxed{\text{Stronger version}}} \mu(A_k) = 0$ or 1

$\mu(f^{-1}(A_k)) = \mu(A_k)$

Due to $\operatorname{supp} \mu = X$. $\Rightarrow \mu(U_k) > 0$, $\mu(A_k) = 1$

$A = \bigcap_{k \geq 1} A_k$ $\mu(A) = 1$

& $\forall x \in A$. $\forall k$, $x \in A_k$, $\exists n$. s.t $f^n(x) \in U_k$. $\square$

Ex. $\mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N$, $\mu = $ Leb. , $\mathcal{B}$

i). $R_\alpha(x) = (x_1 + \alpha_1, \cdots, x_N + \alpha_N)$., preserve Leb

R_α ergodic $\Leftrightarrow$ $1, \alpha_1, \cdots, \alpha_N$ rational independent ii). M integer ≥ 2. $g: \theta \to M\theta$. ergodic 3)

Rk. What about a matrix $A \in M_{N\times N}(\mathbb{Z})$. $\theta \to A\theta$.

Pf: Use fourier coefficients, $\hat{\varphi}(k) = \int_{\mathbb{T}^N} \varphi\, e^{2\pi i k\cdot\theta}\, d\,\mathrm{Leb}(\theta)$
$k \in \mathbb{Z}^N$,

~~1, $\alpha_1, \dots, \alpha_N$ rational dependent.~~

$$\widehat{\varphi\circ R_\alpha}(k) = \int \varphi\circ R_\alpha\, e^{2\pi i k\cdot\theta}\, d\theta$$

$$= \int \varphi\, e^{2\pi i k\cdot(R_\alpha^{-1}\theta)}\, d\theta$$

$$= e^{2\pi i(k\cdot(-\alpha))} \int \hat{\varphi}(k)$$

$$\widehat{\varphi\circ g}(k) = \int \varphi\circ g(\theta)\, e^{2\pi i k\cdot\theta}\, d\theta$$

If $M | k$.
$$= \hat{\varphi}(k/M)$$

i). If φ is a.e. f-inv. $\Rightarrow \forall k$

$$\hat{\varphi}(k) = \widehat{\varphi\circ R_\alpha}(k) = e^{2\pi i(k\cdot(-\alpha))}\cdot \hat{\varphi}(k)$$

① $(1, \alpha)$ rational dependent, $\exists k$. $k\cdot\alpha \in \mathbb{Z}$.

$\hat{\varphi}(k) = 1$, $\rightsquigarrow \varphi: \theta \mapsto e^{-2\pi i(k\cdot\theta)}$, f-inv

② $(1, \alpha)$ rational independent. $\forall k \neq 0$ $k\cdot\alpha \notin \mathbb{Z}$

$\Rightarrow \hat{\varphi}(k) = 0$. $\forall k = 0$. $\Rightarrow \varphi = \text{cst}$ a.e

ii) $\hat{\varphi}(k) = \hat{\varphi}(k\cdot M) = \hat{\varphi}(k\cdot M^2) \dots$

$\varphi \in L^2$, $\hat{\varphi} \in L^2$ $\Rightarrow \hat{\varphi}(k) = 0$. $\forall k \neq 0$

Ex 2. $(\Lambda^{\mathbb{N}}, \sigma)$. see exercise.

$\sigma^{-1}(E)=E$, $\exists A$ finite union of cylinders. $\mu(E \triangle A) < \varepsilon$. $\mu(E) = \mu(\sigma^{-n_0}(E) \cap E)$

$\exists n_0$, $\mu(\sigma^{-n_0}(A) \cap A) = \mu(A)^2$, $\Rightarrow$ $\sim \mu(E)^2$

二 Ergodic theorem

Thm. (Birkhoff). $(X, \mathcal{B}, \mu)$. μ, f.

μ is f-inv

$\forall \varphi \in L^1(X, \mu)$. Write

$$S_n \varphi(x) = \frac{1}{n} \sum_{0 \le j \le n-1} \varphi \circ f^j(x).$$

(i). The limit $\hat{\varphi}(x) = \lim_{n\to\infty} S_n \varphi(x)$ exists a.e. μ

(ii) $\hat{\varphi} \circ f(x) = \hat{\varphi}(x)$. a.e. μ

(iii) $\|\hat{\varphi}\|_{L^1} \le \|\varphi\|_{L^1}$

(iv). μ finite. L^1-convergence

$$\|S_n \varphi - \hat{\varphi}\|_{L^1} \to 0$$

(v) $\forall A \in \mathcal{B}$, $\mu(A) < \infty$, $f^{-1}(A) = A$. then

$$\int_A \varphi \, d\mu = \int_A \hat{\varphi} \, d\mu$$

(vi) If μ ergodic probability, then

$$\hat{\varphi}(x) = \int \varphi \, d\mu. \quad \text{a.e.}$$

$S_n \varphi$. called Birkhoff average.

$\lim S_n \varphi$. called ~~tem~~ time average, orbit average

$\int_X \varphi \, d\mu$ space average

<u>Cor</u>: $A \in \mathcal{B}$, $\varphi = 1_A$

$$S_n\varphi(x) = \frac{1}{n} \# \{ j \leq n \mid f^j(x) \in A \}$$

μ prob. ergodic $\quad S_n\varphi(x) \longrightarrow \mu(A)$ a.e.

<u>Cor</u>. a.e. $x \in [0,1[$ $\quad \forall\ j \in \{0,1,\dots,9\}$.
$x = 0.x_1x_2\cdots$

$$\lim_{n\to\infty} \frac{1}{n} \# \{ k \leq n \mid x_k = j \} = 1/10$$

<u>Pf</u>. $[0,1[\simeq \mathbb{R}/\mathbb{Z}$. $\quad \forall j \to I_j \subset \mathbb{R}/\mathbb{Z}$, $\mu(I_j) = \frac{1}{10}$

$f : x \to 10x$. $\quad \mu = $ Leb. $-$ f-ergodic $\square$

$$\{ k \leq n \mid f^k(x) \in I_j \} = \{ k \leq N \mid x_k = j \}$$

<u>Pf. ergodic thm</u>..

<u>Lem</u>: (Maximal ergodic thm)

$\varphi \in L^1(X,\mu,\mathbb{R})$. $\quad \varphi^*(x) = \cancel{\max_k} \sup_{n\geq 1} \cancel{n} S_n\varphi(x)$

then
$$\int_{\{\varphi^*(x) > 0\}} \varphi \, d\mu \geq 0$$

<u>Pf</u>: (Tricky). Let $\quad \psi_n = \max\{0, \varphi, \varphi\circ f + \varphi, \dots, \varphi\circ f^n + \dots + \varphi\}$

On $\{\psi_n > 0\}$, we have

$$\psi_{n-1}\circ f + \varphi \geq \max_{0\leq k \leq n} \{ \varphi\circ f^k + \dots + \varphi \}$$

due to $\psi_n(x) > 0$ $\quad \geq \max_{-1 \leq k \leq n} \{ \varphi\circ f^k + \dots + \varphi \} = \psi_n$

Therefore

$$\int_{\{\psi_n>0\}} \varphi \geq \int_{\{\psi_n>0\}} \psi_n - \int_{\{\psi_n>0\}} \psi_{n-1}\circ f$$

$$= \int_X \psi_n - \int_{\{\psi_n>0\}} \psi_{n-1}\circ f$$

(due to $\psi_{n-1}\circ f \geq 0$) $\geq \int_X \psi_n - \int_X \psi_{n-1}\circ f = \int_X \psi_n - \psi_{n-1}\circ f \geq 0$

$\psi_n \geq \psi_{n-1}\circ f$.

Now $\{\varphi^* > 0\} = \bigcup \{\psi_n > 0\}$, $\{\psi_n > 0\} \subset \{\psi_{n+1} > 0\}$

$$\Rightarrow \int_{\{\varphi^*>0\}} \varphi \geq 0$$

(Back to Pf of Birkhoff ergodic thm), $\forall \ \alpha < \beta$

Let $E_{\alpha,\beta} = \{x \mid \liminf S_n\varphi(x) < \alpha < \beta < \limsup S_n\varphi(x)\}$

Notice $f^{-1}E_{\alpha,\beta} = E_{\alpha,\beta}$. Apply lem with $X = E_{\alpha,\beta}$

Due to $\{(\varphi-\beta)^* > 0\} \cap E_{\alpha,\beta} = E_{\alpha,\beta}$,

$$\int_{E_{\alpha,\beta}} \varphi - \beta = \int_{E_{\alpha,\beta}\cap\{(\varphi-\beta)^*>0\}} \varphi - \beta \geq 0$$

$$\int_{E_{\alpha,\beta}} \varphi \geq \beta\,\mu(E_{\alpha,\beta})$$

Consider $-\varphi + \alpha$, similarily, $\int_{E_{\alpha,\beta}} \varphi \leq \alpha\,\mu(E_{\alpha,\beta})$

Hence $\mu(E_{\alpha,\beta}) = 0$. $\forall\ \alpha,\beta$

Therefore a.e. x, $\liminf S_n\varphi(x) = \limsup S_n\varphi(x)$ $\square$

(ii) $\tilde{\varphi} \circ f = \lim S_n (\varphi \circ f)$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} S_{n+1}\varphi - \frac{1}{n}\varphi \right) \cancel{S_n\varphi} = \tilde{\varphi} \quad \text{a.e.}$$

(iii) ~~$\forall N \quad \varphi_N = \min\{|\varphi|, N\}$~~

Suppose $\varphi \geq 0$. then Fatou lem.

$$\int \tilde{\varphi} = \int \lim S_n \varphi \leq \liminf \int S_n \varphi = \int \varphi$$

(iv) Use dominated convergence. , $\varphi_N^{*} = \begin{cases} \min\{\varphi, N\} & \varphi \geq 0 \\ \max\{\varphi, -N\} & \varphi < 0 \end{cases}$

$$|\varphi - \varphi_N|_{L^1} \xrightarrow{\text{as } N\to\infty} 0.$$

$$\| S_n\varphi_N - \tilde{\varphi}_N \|_{L^1} \xrightarrow{N\to\infty} 0 \quad \text{due dominated convergene}$$

$$\| S_n(\varphi - \varphi_N) \|_{L^1} \leq \| \varphi - \varphi_N \|_{L^1} \to 0$$

$$\| \tilde{\varphi}_N - \tilde{\varphi} \|_{L^1} = \| \widetilde{(\varphi_N - \varphi)} \|_{L^1} \leq \| \varphi_N - \varphi \|_{L^1} \longrightarrow 0$$

(v) Due to $f^{-1}(A) = A$. $\int_A \varphi \circ f \, d\mu = \int_A \varphi \, d\mu$

Hence

$$\int_A \varphi \, d\mu = \int_A S_n \varphi \, d\mu \underset{(iv)}{\xrightarrow{n\to\infty}} \int_A \tilde{\varphi} \, d\mu$$

(vi) (ii) + ergodic $\Rightarrow$ $\tilde{\varphi} = \text{cst}$ a.e.

$$\tilde{\varphi}(x) = \int \tilde{\varphi} = \lim \int S_n \varphi = \int \varphi. \quad \square$$

$(\Omega, \mathcal{A}, P)$ probability space. $(\mathbb{R}, \mathcal{B})$. random variable. $X_i : (\Omega, \mathcal{A}, P) \to (\mathbb{R}, \mathcal{B})$

Example: (law of large numbers)

Let $X_1, X_2, X_3 \ldots$ be a sequence of independent, identical distributed real random variables with finite expectation.

Then a.e. $\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \longrightarrow E(X_1)$.

Add proof of ergodicity of $\mu^{\otimes \mathbb{N}}$ Here

Let μ be the distribution of X_1, $\mu = (X_1)_*(\mathbb{P})$

$$\mu(A) = \mathbb{P}(\omega;\ X_1(\omega) \in A) \qquad \forall A \in \mathcal{B} = \mathcal{B}(\mathbb{R})$$

Independent. $A_{k_i} \in \mathcal{B}$

$$\mathbb{P}(X_{k_i} \in A_{k_i},\ k_i \text{ finite}) = \prod \mathbb{P}(X_{k_i} \in A_{k_i})$$

Pf. Construct a space. $(X = \mathbb{R}^{\mathbb{N}}, \mathcal{B}, \mu^{\otimes \mathbb{N}})$ σ shift

$$\varphi : X \to \mathbb{R}, \quad (x_1, x_2, \cdots) \mapsto x_1$$

Distribution of $(X_1, X_2, X_3, \cdots)$ (independence)

identical to ~~$(x_1, x_2, \cdots)$ with~~ measure $\mu^{\otimes \mathbb{N}}$.

$v : \Omega \to \mathbb{R}^{\mathbb{N}}$, $\quad v(\omega) = (X_1(\omega), X_2(\omega), \cdots)$. $\quad (v)_*P = \mu^{\otimes \mathbb{N}}$

$$\frac{1}{n}(X_1 + X_2 + \cdots + X_n)(\omega) = S_n\varphi(X_1^{(\omega)}, X_2^{(\omega)}, X_3^{(\omega)}, \cdots) = S_n \varphi \circ v(\omega)$$

(Exercise, $\mu^{\otimes \mathbb{N}}$ is σ-ergodic). By Birkhoff ergodic thm,

$$\text{a.e.} \quad S_n\varphi \longrightarrow \tilde{\varphi} = \int x \, d\mu = \mathbb{E}(X_1).$$

$$\begin{tikzcd}
(\Omega, \mathcal{A}, P) \arrow[r, "\sigma"] \arrow[d, "v:"', "\text{semi-conjugate}"] & (\Omega, \mathcal{A}, P) \arrow[d, "v"] \\
(\mathbb{R}^{\mathbb{N}}, \mu^{\otimes \mathbb{N}}) \arrow[r, "\sigma"] & (\mathbb{R}^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})
\end{tikzcd}$$

$$K = \{\omega \mid \lim \tfrac{1}{n}(X_1(\omega) + \cdots + X_n(\omega)) \to E(X_1)\}$$

$$K' = \{x \mid \lim \tfrac{1}{n}(\varphi(x) + \varphi(\sigma(x)) + \cdots + \varphi(\sigma^{n-1}(x))) \to E(X_1)\}$$

$$v^{-1}(K') = K$$

Thm . von Neumann ergodic thm

Let $(X, \mathcal{B}, \mu)$ probability space. $f \circlearrowright X$ preserve μ.

Then $\forall \varphi \in L^2(X, \mu)$

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ f^k = P\varphi \quad \text{in } L^2. \quad (\star)$$

where P the projection from $L^2(X, \mu)$ to the subspace of f-inv functions.

Pf: Let $\mathcal{H} \subseteq L^2(X, \mu)$, f-inv subspace. $U\varphi = \varphi \circ f$.

then due to ~~f~~ μ is f-inv. $|U\varphi|_{L^2} = |\varphi|_{L^2}$.

U is an isometry ~~unitary operator~~.

Step 1: If $\varphi = \psi - U\psi$, then true. $(\star)$

Pf. $\frac{1}{n} \sum_{k=0}^{n-1} U^k\varphi = \frac{1}{n} \sum_{k=0}^{n-1} (U^k\psi - U^{k+1}\psi) = \frac{1}{n}(\psi - U^n\psi) \to 0$

and, $\varphi \in \mathcal{H}^\perp$. so $P\varphi = 0$

Step 2: $L = \{\psi - U\psi, \psi \in L^2\}$, then $\bar{L} = \mathcal{H}^\perp$.

Pf. $L \subset \mathcal{H}^\perp$. then $\bar{L} \subset \mathcal{H}^\perp$.

If $\bar{L} \subsetneq \mathcal{H}^\perp$. $\exists\, 0 \neq \varphi \in \mathcal{H}^\perp \cap L^\perp$,

~~But~~ The $\forall \psi \in L^2$ $\langle \varphi, \psi - U\psi \rangle = 0$, U^* adjoint ope

$\Rightarrow \langle \varphi - U^*\varphi, \psi \rangle = 0 \Rightarrow \varphi = U^*\varphi$,

Consider $\|U\varphi - \varphi\|^2 = |\varphi|^2 - 2\langle \varphi, U\varphi\rangle + |U\varphi|^2 = 2|\varphi|^2 - 2\langle U^*\varphi, \varphi\rangle$

Hence $U\varphi = \varphi \Rightarrow \varphi \in \mathcal{H}$. Contradiction to $0 \neq \varphi \in \mathcal{H}^\perp$.

Step 3. If $\varphi_n \to \varphi$. and φ_n satisfies ($\star$)

then φ also

$$\left| \left(\frac{1}{n} \sum_{k=0}^{n-1} U^k \varphi \right) - \left(\frac{1}{n} \sum_{k=0}^{n-1} U^k \varphi_m \right) \right|$$

$$\leq \left| \frac{1}{n} \sum_{k=0}^{n-1} U^k (\varphi - \varphi_m) \right| \leq | \varphi - \varphi_m | \to 0 \qquad \square$$

~~Invariant measures and ergodic measures~~

Rk. 1. L^2- convergence generalizes to amenable group with no difficulty. (Følner sequence)

2. Birkhoff ergodic thm (pointwise ergodic thm) is harder. The general strategy

Step 1. Maximal ergodic thm. estimate

Step 2. Find a dense subspace, where the convergence is true.

ex. (~~finite~~ Amenable gp

See for example:
Ben Krause POINTWISE ERGODIC THEORY: EXAMPLES AND ENTROPY

ex. polynomial ergodic thm (Bourgain)

$(X, \mathcal{B}, \mu) \circlearrowleft f$, p integer coefficient poly.

$\forall\ \varphi \in L^r(X, \mu)$, $r > 1$

a.e. x. $\lim \frac{1}{N} \sum_{0 \leq n \leq N-1} \varphi(f^{p(n)}(x))$. exists.

3. What about convergence at every pt ? (unique ergodicity)

4. Rate to the average. ?

See for example: Kesten, Uniform distribution mod 1 (II) for circle rotation
Forni, Asymptotic Behaviour of Ergodic Integrals of 'Renormalizable' Parabolic Flows.
The introduction and the works mentioned.

Invariant measures and ergodic measures

Thm (Banach-~~Anagolu~~ Alaoglu). Let X be a cpt metric space, $\mathcal{B}$ the Borel σ-algebra. The space of Borel probabality measure $\mathcal{M}(X)$ is weak* cocompact.

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i.e. $\forall$ ~~$f \in C(X)$~~, sequence $\{\mu_n\} \subset \mathcal{M}(X)$

$\exists$ a convergent subsequence $\{\mu_{k_n}\}$ to $\mu \in \mathcal{M}(X)$

i.e. convergence : $\forall f \in C(X)$, $\lim \int f \, d\mu_{k_n} = \int f \, d\mu$.

Thm. Let X be a cpt metric space, $(X, \mathcal{B}) \circlearrowleft f$ continuous. Then $\exists$ f-inv Borel measure.

Pf. Take $x \in X$. consider Cesaro average

$$\mu_N = \frac{1}{N} \sum_{0 \le n \le N-1} \delta_{f^n(x)} ,$$

By Banach-~~Anagolu~~ Alaoglu, $\exists$ μ, $\{k_n\}$. $\mu_{k_n} \to \mu$

$$\begin{aligned} f_* \mu - \mu &= \lim f_* \mu_{k_n} - \mu_{k_n} \\ &= \lim \frac{1}{k_n} \left(\delta_{f^{k_n+1}(x)} - \delta_x \right) \to 0 \end{aligned}$$

Hence $f_* \mu = \mu$. thus μ is f-inv Borel measure.

Rk. (1) Gives another proof a Birkhoff recurrence, without the use of Axiom of choice.

(2). In general, the invariant measure not unique

Ergodic measures. $\mathcal{M}(X)$ is a convex set.

~~Milman~~,

extreme point of $\mathcal{M}(X)$: If $\mu = c\mu_1 + (1-c)\mu_2$

with $1 > c > 0$. and $\mu_1, \mu_2 \in \mathcal{M}(X)$, then $\mu_1 = \mu_2$

Krein-Milman: ~~every point in $\mathcal{M}(X)$~~ ;

$\mathcal{M}(X)$ = convex hull of extreme points

Thm. Let $\mathcal{M}_1(X) \subset \mathcal{M}(X)$. subset of f-inv measures

Then f-ergodic measures $\Longleftrightarrow$ extreme pt of $\mathcal{M}_1(X)$.

Pf. If μ is f-ergodic, suppose $\mu = c\mu_1 + (1-c)\mu_2$

By Radon-Nykodym thm $\exists$ density function φ_1, φ_2

s.t $d\mu_1 = \varphi_1 d\mu$, $d\mu_2 = \varphi_2 d\mu_2$

μ, μ_1, μ_2 f-inv $\Rightarrow$ φ_i is μ a.e. f-inv.

ergodicity $\Rightarrow$ $\varphi_i = cst$ a.e.

μ, μ_1, μ_2 probability $\Rightarrow$ $\varphi_i = 1$ a.e. $\Rightarrow$ $\mu_1 = \mu_2 = \mu$

If μ - extreme pt, $\forall A \in \mathcal{B}$. f-inv

$\mu_1 = \frac{1}{\mu(A)} \mu|_A$, $\mu_2 = \frac{1}{1-\mu(A)} \mu|_{A^c}$, $\mu = \mu(A)\mu_1 + (1-\mu(A))\mu_2$

If. $0 < \mu(A) < 1$, $\Rightarrow$ $\mu_1 = \mu_2$, but $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \emptyset$. $\Box$