(一) Poincare recurrence

Thm $(X, \mathcal{B}, \mu) \circlearrowleft f$. measure preserving.

μ. probability measure

$\forall A \in \mathcal{B}$. a.e $x \in A$. $(f^n(x))_{n\in\mathbb{N}}$ return to A infinitely many times

Pf. Def $B_n = \{ x \in A \mid f^m(x) \notin A,\ \forall\, m \geq n \}$.

Let $g = f^n$; $B = B_n$, g preserve μ. & $\forall\, x \in B$, $g^l x \notin A$.

We have $g^{-l}B \cap B = \phi$. since $\forall\, x \in g^{-l}B \cap B$

$g^l x \in B \subset A$, & $x \in B$. contradicts to def of B.

Hence $B, g^{-1}B, g^{-2}B, \cdots,$ disjoint $\subset X$.

but $\mu(g^{-l}B) = \mu(B)$.

$$1 \geq \mu\Big(\bigcup_{l\geq 0} g^{-l}B\Big) = \sum_{l\geq 0} \mu(g^{-l}B) = \sum_{l\geq 0} \mu(B).$$

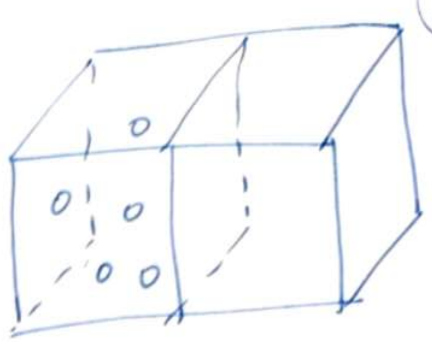
$\Rightarrow \mu(B) = 0$

$\forall\, n \quad \mu(B_n) = 0 \quad \Rightarrow \mu(A \setminus \bigcup_n B_n) = \mu(A)$

$\forall\, x \in A \setminus \bigcup_n B_n$. recurrent.

Ex: (Zermelo paradox)

2 box of size 1.



If all points in first box.

$A = \{$ ~~neighborhood of initial~~ all points in first box $\}$.

Poincare recurrence, Leb a.e. $x \in A$, $(f^t(x))_{t \geq 0}$ return to A, paradox to statistic mechanics

Time to return $2^{10^{23}}$ .... too large to observe

二) Birkhoff recurrence.

Thm: X cpt metric space, $f \circlearrowleft X$. continuous. Then $\exists\ x \in X$. s.t. $\exists\ n_k \to \infty$, $\lim_{k\to\infty} f^{n_k}(x) = x$.

Def. Such x called recurrent pt. (compared to periodic pt)

Remark: 1) If X not cpt, not true

$X = \mathbb{R}$. $f(x) = x+1$

2). May have only one recurrent pt

$X = \mathbb{R} \cup \{\infty\}$. $f(x) = x+1$, $x = \infty$ recurrent pt

3). May have no periodic pt, irrational rotations

Def (Minimal set).

f. invariant set A : $f(A) \subset A$.

A closed non-empty invariant set is minimal if there exist no smaller such set in A.

Prop. If Y a minimal set in X, then

$\forall\ y \in Y$. $\overline{\{f^n(y), n \geq 0\}} = Y$, recurrent.

Pf. $\overline{\{f^n(y), n \geq 0\}}$ closed, invariant $\subset Y$.

Prop. X cpt metric space. $f \circlearrowleft X$ continuous

Then $\exists$ minimal set

Pf. $\mathcal{F} = \{$ non-empty closed, invariant set $\subset X \}$.

$\mathcal{F}$ partially ordered.

$\forall$ descending sequence. $F_n \supset F_{n+1} \supset \cdots$.

$\bigcap F_n \in \mathcal{F}$, $\bigcap F_n \neq \emptyset$ due to compact

Zorn lemma: $\exists$ a minimal element in $\mathcal{F}$.

(equivalent to Axiom of choice)

Cor. Birkhoff recurrence

Ex. 1. R_α. α circle rotation

$\forall\, x \in S^1 = \mathbb{R}/\mathbb{Z}$. recurrent for R_α.

Pf. Birkhoff recurrence $\Rightarrow \exists\ y \in S^1$, recurrent.

$\forall\, x \in S^1$, $\exists\ \beta \in \mathbb{R}$. s.t. $x = R_\beta y$.

$\lim R_\alpha^{n_k}(y) = y$.

$\lim R_\alpha^{n_k}(x) = \lim R_\alpha^{n_k} R_\beta(y) = \lim R_\beta R_\alpha^{n_k} y$

$= R_\beta y = x$.

~~(transitive commut~~

Ex 2. $(\Lambda^{\mathbb{N}}, \sigma)$ or $(\Lambda^{\mathbb{Z}}, \sigma)$.

三) Multiple recurrence.

Thm (Furstenberg - Weiss)

$f \circlearrowright X$. cpt metric space. $\forall\, l$.

$\exists\ x$, n_k. s.t. $\forall\, 1 \le j \le l$.

$$\lim_{k\to\infty} f^{jn_k}(x) = x$$

$\exists$ recurrent pt $(x, \cdots, x)$ for $(f, f^2, \cdots, f^l)$

Thm (Application) Van der Waerden

$\forall$ finite partion $B_1, \cdots, B_q$ of $\mathbb{N}$.

~~$\forall \ell$. $\exists$ length ℓ arithemetic sequence in~~

For some B_j. $\exists$ arbitrary long arithemetic sequence

Pf. $\Lambda = \{1, \cdots, q\}$. $\Lambda^{\mathbb{N}}$.

$\omega \in \Lambda^{\mathbb{N}}$, $\omega_i = j$ if $i \in B_j$.

$X = \overline{\{\sigma^n(\omega), n \in \mathbb{N}\}} \subset \Lambda^{\mathbb{N}}$.

(X, σ). Furstenberg-Weiss

$\Rightarrow \forall \ell$. $\exists$ $x \in X$. s.t. $\lim \sigma^{jn_k} x = x$ $\forall$ $1 \le j \le \ell$

Take n_k. $d(\sigma^{jn_k} x, x) < 1/10$. $\forall$ $1 \le j \le \ell$

$x = (x_0, x_1, x_2, \cdots)$

$\Rightarrow$ $x_0 = x_{n_k} = \cdots = x_{\ell n_k}$

By $x \in \overline{\{\sigma^n(\omega), n \in \mathbb{N}\}}$

$\exists$ m, $d(x, \sigma^m \omega) < 1/2^{\ell n_k + 1}$

$\Rightarrow$ $\omega_{m+jn_k} = x_{jn_k} = x_0$.

$\Rightarrow$ $\{m, m+n_k, \cdots, m+jn_k\} \subset B_{x_0}$

~~Bet.~~ Bergelson.

Pf. (Furstenberg - Weiss).

$l = 1$ ✓ Birkhoff recurrence

If $l-1$ true, consider l, Assume $f \circlearrowleft X$ minimal

$X^l \supset \Delta = \{(x, x, \cdots, x) \mid x \in X\}$ diag

$g = (f, f^2, \cdots, f^l)$

$h = (f, f, \cdots, f)$

. $h \circlearrowleft \Delta$ minimal, $h \circ g = g \circ h$

Step 1. $\exists$ $y_0 \in \Delta$, $z_{0,k} \in \Delta$ s.t

$$\lim_{k \to +\infty} g^{n_k}(z_{0,k}) = y_0$$

Pf. Take x. $\lim f^{j n_k}(x) = x$ $\forall\ 1 \le j \le l-1$.

(Induction hypothesis)

$y_0 = (x, \cdots, x)$. $z_{0,k} = (x_k, \cdots, x_k)$

where $f^{n_k} x_k = x$ $\left(\begin{array}{l} f(X) \subset X,\ X \text{ minimal} \Rightarrow f(X) = X \\ f \text{ surjective. } \exists\ x_k \end{array} \right)$

$g^{n_k}(x_k, \cdots, x_k) = (x, f^{n_k}(x), \cdots, f^{(l-1)n_k}(x)) \to (x, \cdots, x)$
$= y_0$

Step 2. $\forall y \in \Delta$, $\forall \varepsilon > 0$. $\exists$ $z \in \Delta$, $n \ge 1$

$d(g^n z, y) < \varepsilon$ (image of g large)

Pf. By $h \circlearrowleft \Delta$ minimal $\exists m$

s.t $d(h^m y_0, y) < \varepsilon/2$

$z = h^m z_{0,k}$, $g^{n_k} z = g^{n_k} h^m z_{0,k}$

$= h^m g^{n_k} z_{0,k}$

$\longrightarrow h^m y_0$

Take n_k, s.t. $d(h^m g^{n_k} z_{0,k}, h^m y_0) < \varepsilon/2$.

Step 3 $\forall \varepsilon > 0$. $\exists y$, $n \geq 1$. s.t

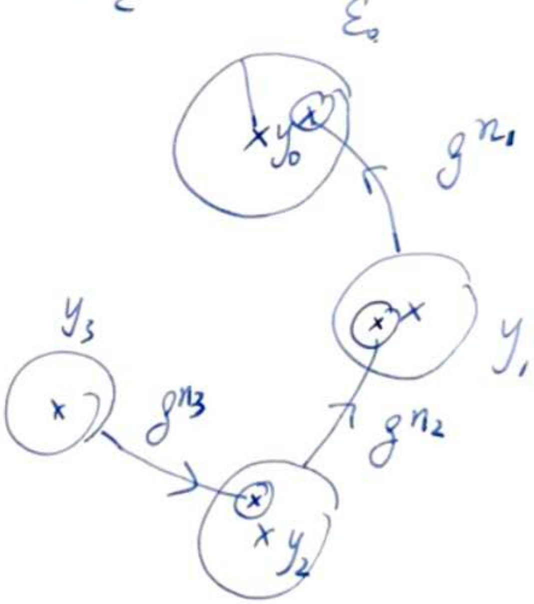
$$d(g^n y, y) < \varepsilon$$

Pf: take y_0, $\varepsilon_0 = \varepsilon/2$

Step2 $\Rightarrow y_1$. $\varepsilon_1 < \varepsilon_0$

$\forall z \in B(y_1, \varepsilon_1)$

$d(g^{n_1} z, y_0) < \varepsilon_0$



Induction

y_k, $\varepsilon_k < \varepsilon_{k-1}$

$\forall z \in B(y_k, \varepsilon_k)$ $d(g^{n_k} z, y_{k-1}) < \varepsilon_{k-1}$.

$\forall j > i$.

$$d(g^{n_{i+1} + \cdots + n_j}(y_j), y_i) < \varepsilon_i < \varepsilon/2$$

By compactity of Δ. $\exists\ i, j$ $d(y_i, y_j) < \frac{\varepsilon}{2}$. 8

(A sequence in a compact metric space, $\exists$ accumulation pt)

$\Rightarrow$

$$d(g^{n_{i+1} + \cdots + n_j}(y_j), y_j) < \varepsilon.$$

Step 4. $\varphi(z) = \inf_{n \geq 1} d(g^n z, z)$.

~~φ~~ Goal: Find x. s.t. $\varphi(x) = 0$

$\underline{\varphi(z) \quad \text{upper semi-continuous}}$ $\quad \varlimsup_{y_n \to y} \varphi(y_n) \leq \varphi(y)$

Continuity pt of φ dense in Δ.

a continuous pt of. φ. If $\varphi(a) > 0$, $\exists\ \delta$.

$\exists\ V \overset{\text{open}}{\subset} \{x \mid \varphi(x) > \delta\}$ ~~open~~

h minimal. $\Rightarrow$ $\bigcup_{m \geq 1} h^{-m}(V) = \Delta$

Δ cpt $\exists$ finite set F. $\bigcup_{m \in F} h^{-m}(V) = \Delta$.

Take y Step 3. $y \in h^{-m}(V)$. $m \in F$

$(\star) \Rightarrow$ $d(g^n y, y) < \varepsilon$ & $d(g^n h^m y, h^m y) > \delta$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$

$$\qquad\qquad\qquad\qquad\qquad\qquad d(h^m g^n y, h^m y).$$

Take ε small s.t. h^m uniform continuous $\forall m \in F$

that is $\forall d(y_i, y_j) < \varepsilon$, $\forall m \in F$, $d(h^m y_i, h^m y_j) < \delta$.

Contradiction to $(\star)$