Decay of coefficients and Mixing

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Thm. Let Γ be a lattice of $SL_N(\mathbb{R})$ and $g \in G$ s.t. $\lim_{n\to\infty} \|g^n\| = \infty$. Then g on $\Gamma\backslash G$ is mixing ~~w.r.t.~~ for the measure μ

Pf: Due to $\mathcal{H} = L^2_0(\Gamma\backslash G, \mu)$ has no non-trivial G-inv. vector. it suffices to prove

Thm (Howe - Moore) Let $G = SL_N(\mathbb{R})$, π a unitary rep of G on a Hilbert space $\mathcal{H}$., and $v, w \in \mathcal{H}$. If $\mathcal{H}^G = \{0\}$, then

$$\lim_{\|g\|\to\infty} (\pi(g)v, w) = 0$$

We need some preparation

Lem. Cartan decomposition; Let $K = SO_N(\mathbb{R})$, $A = \text{diag}$. $= \left\{\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{pmatrix}, a\right.$

Then $G = KAK$

Pf: $\forall g \in G$, $g^t g$ is a symmetric matrix.

$\exists\, k \in SO_N$, s.t. $g^t g = k^t \cancel{\text{diag}}\, b\, k$, $b \in A$

$b = \text{diag}\{b_i\}$ $b_i > 0$

We can find $a \in A$ s.t. $a^2 = \cancel{diag}\, b$ ②

then $h = g k^{-1} a^{-1}$. we have

$$h^t h = a^{-t} k^{-t} g^t g k^{-1} a^{-1}$$
$$= a^{-t} b a^{-1} = id$$

Hence $h \in K$. then $g = \cancel{kah}\ hak \in KAK$ □

Rm. A^+.

Lem. (Weak compactness.) If $\mathcal{H}$ is separable Let $\{v_n\}$ be a sequence of bounded vectors in $\mathcal{H}$. then $\exists$ subsequence v_{j_n} and $v \in \mathcal{H}$. s.t. $v_{j_n} \xrightarrow{weak} v$.

i.e. $\forall w \in \mathcal{H}$ $\langle v_{j_n}, w \rangle \to \langle v, w \rangle$

Pf: Since $\mathcal{H}$ is separable. let $\{w_n\}$ be a countable dense subset of $\mathcal{H}$. By a diagonal argument ~~for each w_n~~ We can find a subsequen v_{j_n} s.t $\forall w_m$, we have $\langle v_{j_n}, w_m \rangle$ converge $j_n \to \infty$

Let $f(w_m) = \lim \langle v_{j_n}, w_m \rangle$. By density of $\{w_n\}$ Then f ~~is a~~ extends to bounded linear operator on $\mathcal{H}$

By Reisz rep, $\exists\ v \in \mathcal{H}$. s.t $f(\ ) = \langle v, \cdot \rangle$ □

Pf of Howe-Moore

By restriction to the subspace generated by v, w.

we can suppose the whole space $\mathcal{H}$ is separable

Suppose $\|g_n\| \to \infty$ and $(\pi(g_n)v, w) \not\to 0$

By Cartan decomposition $g_n = h_n a_n k_n \in KAK$

By compactness of K, find convergent subsequence of h_n, k_n

by continuity of unitary representation,

it suffices to consider

$$(\pi(a_n)v, w) \not\to 0 \qquad \|a_n\| \to \infty$$

Notation.

Let $E_{ij}(u) = Id + u \begin{array}{c} j \\ \left(\cdots 1 \right) \end{array} i$. i-row, j-column

$$A_{ij}(t) = \operatorname{diag}(1, \dots, 1, \underset{i}{t}, 1 \dots 1, \underset{j}{t^{-1}}, 1 \dots, 1)$$

Then $\langle A_{ij}(t), E_{ij}(u), E_{ji}(s) \rangle$ Generate a subgp of $SL_N(\mathbb{R})$ isomorphic to $SL_2(\mathbb{R})$

Since $\|a_n\| \to \infty$, $\exists$ subsequence, $\exists\ i<j$, s.t. $a_n = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_N \end{pmatrix}$.

with $b_i / b_j \to \infty$. (or $b_j/b_i \to \infty$, similar)

Then $$a_n E_{ij}\left(u \frac{b_j}{b_i}\right) = E_{ij}(u)\, a_n .$$

by weak compactness. suppose a subsequence

$$\pi(a_n)v \longrightarrow v_0 \neq 0$$

<u>Claim</u>: v_0 is $E_{ij}(u)$ - inv.

<u>Pf</u>. $\forall\ v' \in \mathcal{H}$.

$$\langle \pi(E_{ij}(u))v_0, v'\rangle = \langle v_0, \pi(E_{ij}(-u))v'\rangle$$

$$= \lim \langle \pi(a_n)v, \pi(E_{ij}(-u))v'\rangle$$

$$= \lim \langle \pi(E_{ij}(u)a_n))v, v'\rangle$$

$$= \lim \langle \pi(a_n)\pi(E_{ij}(\tfrac{b_j}{b_i}u))v, v'\rangle$$

Since $u\,b_j/b_i \longrightarrow 0$ as $n \to \infty$, by continuity

$$= \lim \langle \pi(a_n)v, v'\rangle = \langle v_0, v'\rangle$$

Hence $v_0 = \pi(E_{ij}(u))v_0$ $\square$

Then we can apply proposition of ergodicity to obtain $G_{v_0} > \langle A_{ij}(t), E_{ij}(u), E_{ji}(s)\rangle$.

We can apply Mautner. for $k \neq j$

$$A_{ij}(t)^{-1} E_{ik}(u) A_{ij}(t) = E_{ik}(ut^{-1}) \xrightarrow{t \to +\infty} \text{Id}$$

Hence $E_{ik}(u) \in G_{v_0}$, Similarily $E_{kj}(u) \in G_{v_0}$

Apply again proposition of ergodicity.

$$G_{v_0} > \langle A_{kl}(t), \ \forall\, 1 \le k, l \le N \rangle$$

Notice that. $SL_N(\mathbb{R}) = \langle A_{ij}(t), E_{ij}(u). \ \forall\, 1 \le i \ne j \le n \rangle$

Hence $G_{v_0} = SL_N(\mathbb{R})$. contradicts to $v_0 \ne 0$. $\square$

Rk. (Speed of convergence). Property T, Spectral gap of Lapla

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Examples of lattices of $SL_2(\mathbb{R})$, $SL_2(\mathbb{Z})$, hyperbolic surface

Let $\mathbb{H} = \{z = x+iy : x \in \mathbb{R}, y > 0\}$ be the complex upper plane. For any $g \in SL_2(\mathbb{R})$ and $z \in \mathbb{H}$, we define the fractional linear action by

$$g z := \frac{az+b}{cz+d}.$$

Then $\operatorname{Im} gz = \frac{\operatorname{Im} z}{|cz+d|^2} > 0$. the action preserves $\mathbb{H}$.

Moreover, $g = -Id$ acts trivially and

$$(gg')z = g(g'z). \quad \text{it is a gp action}$$

The tangent bundle of $\mathbb{H}$ is $\mathbb{H} \times \mathbb{C}$

the tangent map is given by $g(z, \xi) = \left(\frac{az+b}{cz+d}, \frac{\xi}{(cz+d)^2}\right)$

For a vector $v \overset{=(z,\xi)}{}$ in ~~$z \times \mathbb{C}$~~ the tangent space $T_z\mathbb{H}$

define its length by $\|v\|_z = \dfrac{\|\xi\|}{\operatorname{Im} z}$.

From the tangent action. the length is preserved under the action of $SL_2(\mathbb{R})$. Indeed, it is the hyperbolic metric.

The hyperbolic volumn is

$$d\,vol_{\mathbb{H}^2}(z) = \frac{dx\,dy}{y^2}.$$

preserved by $SL_2(\mathbb{R})$. verify by $g = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$, $g = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$

$g = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

<u>Unit tangent bundle</u>. $T^1\mathbb{H} = \{ v = (z, \xi) \in \mathbb{H} \times \mathbb{C},\ |v|_z = 1 \}$

<u>lem</u>. The application $\Phi : G \to T^1\mathbb{H}$ given by

$g \longmapsto g(i, i)$. is a bijection.

- $\forall\, g, g' \in G$, we have $\Phi(gg') = g\,\Phi(g')$.

We define hyperbolic length $l(c)$ a C^1 curve $c : [a,b] \to \mathbb{H}$

$$l(c) = \int_a^b |c'(t)|_{c(t)}\, dt$$

hyperbolic distance

$$d(w, z) = \inf_{\substack{c \text{ connecting} \\ w,z}} l(c)$$

A geodesic is a curve $c : I \to \mathbb{H}$ with I an interval of $\mathbb{R}$

such that $d(c(s), c(t)) = |s-t|$.

Lem: Geodesics on $\mathbb{H}$ are given by vertical lines and half circles

Def. (Geodesic flow) For $v = (z, \xi) \in T^1\mathbb{H}$, let $c(t)$ ~~geo~~ geodesic

~~$\varphi_t(v)$ be the un.~~

$$\varphi_t(v) = (c(t), c'(t)).$$

Lem. The family $(\varphi_t)_{t\in\mathbb{R}}$ is a one parameter gp of diffeos of $T^1\mathbb{H}$ which commute with the action of G

that is $\varphi_t(gv) = g\,\varphi_t(v)$.

Moreover. if $v = \Phi(g)$, then

$$\varphi_t(v) = \Phi(g\, a_{t/2}) \quad \text{with } a_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}.$$

Pf The group G preserves distance and hence geodesics

$$\varphi_t(v) = \varphi_t(g\Phi(id)) = g\,\varphi_t(\Phi(id))$$

$$\Phi(g a_{t/2}) = g\,\Phi(a_{t/2}).$$

It suffices to verify $\varphi_t(\Phi(id)) = \Phi(a_{t/2})$

$$\underset{\varphi_t(i, i)}{\overset{\parallel}{\varphi_t(\Phi(id))}}$$

$$= (e^t i, e^t i)$$

$\square$

Therefore for any discrete subgp Γ of G

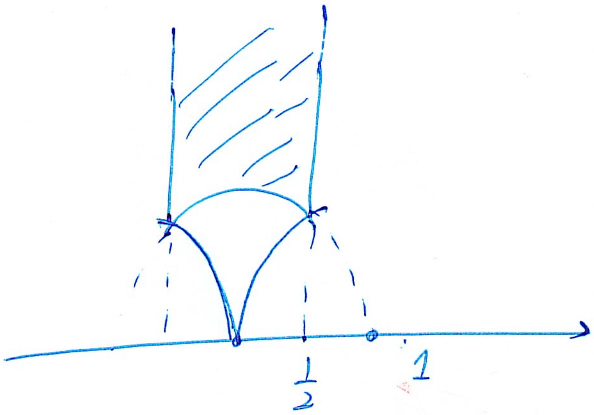
$$\begin{tikzcd} \Gamma\backslash G \arrow[r, "a_{t/2}"] \arrow[d, "\Phi"] & \Gamma\backslash G \arrow[d, "\Phi"] \\ \Gamma\backslash T^1\mathbb{H} \arrow[r, "\varphi_t"] & \Gamma\backslash T^1\mathbb{H} \end{tikzcd}$$

right action

Example. of lattices $\Gamma = SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle$

$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} z = -1/z$

$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} z = z+1$



$D_{2i}(SL_2(\mathbb{Z}))$

$= \{ w \mid d(z,w) \le d(z,\gamma w),\ \forall \gamma \in \Gamma \}$.