成绩 = max{期末成绩, 0.6×期末 + 0.4 (作业, 课堂提问回答)}

~~内容~~:

教材: Peter Walters, An introduction to Ergodic Theory.

Benoist-Paulin, Systèmes dynamiques élémentaires.

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(多提问题, 我学十几年, 会比较熟, 注意不到你们不懂的点.

一 History: PDE, ODE.

Classical mechanics: $x \in \mathbb{R}^n$,

$$\frac{\partial x}{\partial t} = F(x).$$

~~f_t: flow.~~

F continuous: $\mathbb{R}^n \to \mathbb{R}^n$

Cauchy thm: $x(0) = x_0$, $x(t) \triangleq$

~~$f_t(x_0) := x(t)$~~

Def: $f^t(x_0) = x(t)$, flow

- $f^0 = id$.
- $f^t \circ f^s = f^{t+s}$ (ODE solution unique)

$f_t: \mathbb{R}^n \to \mathbb{R}^n$. continuous

(discrete time) long time behavior. Poincaré

stability, dependence on initial value

General dynamics: $f: X \to X$. dynamical system

Topology: positive orbit. $\{x, f(x), f^2(x), \dots, n \geq 0\}$

$$f^n(x) = \underbrace{f \circ f \circ \cdots \circ f}_{n\text{-times}}(x) \qquad (f(x))^n$$

Mordon application. (2)

- Green-Tao (04): $\exists$ arbitrary long arithmetic sequence in prime numbers. (Multiple recurrence)
- Einsiedler-Lindenstrauss-Katok (06): $\{(\alpha,\beta)$ not satisfy Littlewood conj$\}$ very small, Hausdorff dim zero.

Littlewood conjecture: $\forall \alpha, \beta \in \mathbb{R}$

$$\liminf_{n\to\infty} n\{n\alpha\}\{n\beta\} = 0. \qquad \{n\alpha\}$$

decimal part of $n\alpha$

diagonal flow on $SL_3(\mathbb{R})/SL_3(\mathbb{Z})$ $\left(\left(SL_2(\mathbb{R})/SL_2(\mathbb{Z})\right)\right)$

- Rational angle billard, Mirzakani

General language:

- If. f. bijection, orbit $\{f^n(x) \mid n\in\mathbb{Z}\}$.
- $A\subset X$ subset. A -invariant if ~~$f^{-1}(A)=A$~~ $f(A)=A$ (be careful)
- Topological dynamics:
  - X. topological space. f. continuous map.

    f invertible if f homeomorphism

    interested in top property of orbits
- Conjugate: (X,f), (Y,g)

$\exists$ ψ. homeomorphism

s.t

$\forall x \in X$.

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "g"] \\ X \arrow[r, "\psi"'] & Y \end{tikzcd}$$

$\psi \circ f(x) = g \circ \psi(x)$

$\Rightarrow$ same orbit property. f periodic orbits

$\Updownarrow$

g periodic orbits

<u>periodic orbit</u>: If x, $\exists n$. $f^n(x) = x$.

x a periodic pt, n a period of x

<u>dense orbit</u>: x_0, $\{f^n(x), n \geq 0\}$ dense.

$\Updownarrow \psi$

$\{g^n(x), n \geq 0\}$ dense

- <u>semi-conjugate</u>.

$\exists$ ψ. morphism

$g \circ \psi = \psi \circ f$.

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d, "g"] \\ X \arrow[r, "\psi"'] & Y \end{tikzcd}$$

<u>Measurable dynamics</u>: $(X, \mathcal{B}, \mu)$

$\mathcal{B}$ ~~Borel~~ σ-alge, X measurable space

μ measure, $\mathcal{B}$-measurable

- f $\mathcal{B}$-measurable, $\forall A \in \mathcal{B}$ $f^{-1}(A) \in \mathcal{B}$.

μ - a f-invariant measure, if $\forall A \in \mathcal{B}$

$$\mu(f^{-1}(A)) = \mu(A).$$

$(X, \mathcal{B}, \mu, f)$ measurable dynamical system

- Invertible, if f bijection, f^{-1} measurable.

Interested in <u>distribution of orbit</u>

δ_x Dirac measure at x.

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$ probability measure

$\longrightarrow$ $\cancel{M_x}$. m_x some measure

- <u>Conjugate</u>: φ measurable invertible

$$\begin{array}{ccc} (X, \mu) & \xrightarrow{\varphi} & (Y, \nu) \\ \downarrow f & & \downarrow g \\ (X, \mu) & \xrightarrow{\varphi} & (Y, \nu) \end{array}$$

- <u>$\varphi_* \mu = \nu$.</u>

$\varphi_* \mu(A) = \mu(\varphi^{-1}(A))$

- μ. a.e. x. $\varphi \circ f(x) = g \circ \varphi(x)$

same dynamics . . .

三 Criteria of measure preserving

$(X, \mathcal{B}, \mu) \circlearrowleft f$. how to verify $(f)_* \mu = \mu$

<u>Recall</u>: $\mathcal{B}$ σ-algebra. $\forall A \in \mathcal{B}$ subset of X

- $\phi, X \in \mathcal{B}$
- $A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}$
- $A_1, A_2, \cdots \in \mathcal{B} \Rightarrow \bigcup A_i \in \mathcal{B}$

<u>Ex</u>: $[0, 1]$ Borel- σ-algebra

X finite, $\mathcal{B} = \{$ subsets of $X \}$

<u>Recall</u>. Bool algebra $\mathcal{E}$.

- $\phi, X \in \mathcal{E}$
- $A \in \mathcal{E} \Rightarrow A^c \in \mathcal{E}$
- $A_1, A_2 \in \mathcal{E} \Rightarrow A_1 \cap A_2 \in \mathcal{E}$.

<u>Ex</u>. $[0, 1]$, finite union of intervals (

$\bigcup [a_i, b_i)$, $(a_i, b_i]$ $[a_i, b_i]$
(a_i, b_i).

<u>Thm</u>. (Carathéodory)

If $\mathcal{E}$ is a Bool algebra, $\mathcal{B}$ σ-algebra generated by $\mathcal{E}$. If $\mu : \mathcal{E} \to [0, +\infty]$ map

- μ ~~(ϕ)~~ σ-finite. ($X = \cup X_j$. s.t. $\mu(X_j) < +\infty$)
- $E_1, \cdots, E_n \in \mathcal{E}$. $E_i \cap E_j = \phi$

$\Rightarrow \mu(\cup E_i) = \sum \mu(E_i)$

(finite additivity)

(Decreasing) $\forall \{C_n\}$. $\mu(C_0) < \infty$. $C_{i+1} \subset C_i$. $\cap C_i = \phi$. $\Rightarrow \lim_{i \to \infty} \mu(C_i) = 0$

Then $\exists !$ extention of μ, called. μ.

s.t. μ a measure on $\mathcal{B}$.

$\mu(X) = 1$. μ probability measure

<u>Cor 1</u>. $(X, \mathcal{B}, \mu)$, f. If $\mathcal{E}$ Bool algebra generating $\mathcal{B}$, ~~$\Rightarrow$~~ f preserve μ

$\Leftarrow$ $\forall E \in \mathcal{E}$, $\mu(f^{-1}(E)) = \mu(E)$

~~$(X, \mathcal{B}, \mu) \xrightarrow{\psi} (Y, \mathcal{C}, \nu)$~~

$(X, \mathcal{A}) \xrightarrow[f]{} (Y, \mathcal{B}) \xrightarrow[g]{} (Z, \mathcal{B})$

$(g \circ f)_* \mu = g_*(f_* \mu)$ $\quad \mu$ $\mathcal{A}$-measurable

<u>Def.</u> Bool semi algebra

$\boxed{\text{Cor 2}}$ $(X, \mathcal{A}, \mu) \xrightarrow[f]{} (Y, \mathcal{B}, \nu)$

If $\mathcal{E}$ Bool semi-algebra ($\phi \in \mathcal{C}$. $A_1, A_2 \in \mathcal{C}$

$\Rightarrow A_1 \cap A_2$, or A_1^c

finite union of elements of $\mathcal{C}$)

$\mathcal{E} = \{ \bigcup_{\text{finite}} A_j \ , \ A_j \in \mathcal{C} \}$ a Bool algebra) (7)

and $\mathcal{C}$ generate $\mathcal{B}$ as σ-algebra.

Then $f_* \mu = \nu$ if $\forall\ C \in \mathcal{C}$.

$$f_* \mu(C) = \mu(f^{-1}(C)) = \nu(C)$$

Ex. Bool sem-algebra:

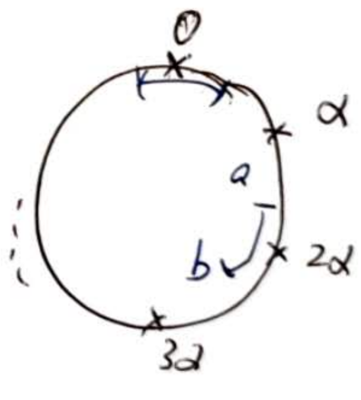
$X = [0, 1]$. $\mathcal{C} = \{ [a,b], (a,b), [a,b), (a,b], \ a \leq b \}$

(VII) Examples.

1. Circle rotation: $S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\{0 \sim 1\}$.

$\forall\ \alpha \in [0,1)$. $R_\alpha x = x + \alpha$, $x \in S^1$.

μ = Leb. measure.



Cor 2. μ is R_α-invariant.

$$(R_\alpha)_* \mu(a,b) = \mu(R_\alpha^{-1}(a,b))$$
$$= \mu((a-\alpha, b-\alpha)) = b-a = \mu(a,b)$$

$\mu(\{a\}) = \mu(\{b\}) = 0.$

2. Symbolic dynamics Bernoulli shift.

$\Lambda = \{1, \dots, q\}$ finite set

$$d(a, b) = \begin{cases} 1 & a \neq b \\ 0 & a = b \end{cases}$$

$\Lambda^{E} = \Lambda^{\mathbb{N}}$ ~~or~~ $\Lambda^{\mathbb{Z}}$. $\quad \omega, \omega'$

$$d(\omega, \omega') = \sum \frac{1}{2^{|i|}} d(\omega_i, \omega_i')$$

$$(\Lambda^{\mathbb{Z}}, d) \overset{\psi}{\simeq} (\text{ Cantor set.}$$

$$\psi(\omega) = \left(\sum_{n \geq 0} \frac{\omega_n}{(q+1)^n}, \sum_{n<0} \frac{\omega_{-n}}{(q+1)^{-n}} \right)$$

- Top dyn: shift σ.

$$\sigma(\omega)_i = \omega_{i+1}.$$

$$(\omega_0, \omega_1, \omega_2, \dots) \xrightarrow{\sigma} (\omega_1, \omega_2, \omega_3, \dots)$$

$$d(\sigma(\omega), \sigma(\omega')) \leq 2 d(\omega, \omega') \Rightarrow$$

σ- is 2- Lip $\Rightarrow$ contin

- Measurable dyn.: Bernoulli measure

A. ν a measure on Λ,

$\mu = \nu^{\otimes \mathbb{Z}}$ measure on $\Lambda^{\otimes \mathbb{Z}}$. verify ν is σ-inv

$\mathcal{B}$ - generated by open sets, disks $B(x, r)$

top base, $B(x, r) =$ cylinder set

$= C_{m, a_0, \dots, a_n} \quad a_i$

$= \{\omega \mid \omega_{m+i} = a_i, \ i = 0, \dots, n\}$

$\{C_m, a_0, \dots, a_n\}$ Borel semi-algebra

Cor 2. only need to verify $C_m, a_0, \dots, a_n$

$$\nu(\sigma^{-1}(C_m, a_0, \dots, a_n)) = \nu(C_{m+1}, a_0, \dots, a_n)$$
$$= \nu(a_0) \cdots \nu(a_n)$$
$$= \nu(C_m, a_0, \dots, a_n).$$

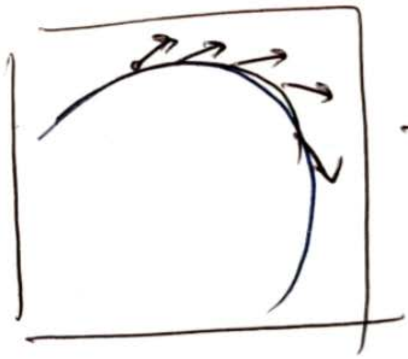
Prop. μ is σ-invariant

3. Liouville measure & Hamiltonian sys.

$\Omega \subset \mathbb{R}^n$ open set, $F: \Omega \to \mathbb{R}^n$. continuous vector field

$\cancel{\partial x}$ $x_0 \in \mathbb{R}^n$

$$\frac{\partial x}{\partial t} = F(x). \quad \text{ODE}$$



Ω

Assump: $\forall\ x_0 \in \Omega$.

ODE solution exists on $\mathbb{R}$

$f^t : \Omega \to \Omega$ flow. $t \in \mathbb{R}$.

Prop (Liouville thm) ρ. density. $\lambda = \rho$ Leb.

λ is f^t flow invariant iff

$$\sum_i \frac{\partial}{\partial x_i}(\rho F_i) = 0$$

$f^0 = id$

Pf. $\Longleftrightarrow \forall\ \varphi \in C_c^\infty(\Omega)$. $\int \varphi \, d\lambda = \int \varphi \circ f_t \, d\lambda$.

$\Omega \to \mathbb{R}$ $\quad \int \varphi \circ f_{s+t} = \int (\varphi \circ f_s) \circ f_t$.

$\Leftarrow\!\!>\forall\varphi.\ \frac{\partial}{\partial t}\int \varphi\circ f_t \, d\lambda\Big|_{t=0} = 0$ φ's still in $C_0^\infty(\Omega,\mathbb{R})$ (10)

$$= \int d\varphi \cdot \frac{\partial}{\partial t} f_t \; \rho\, d(\text{Leb}) \; \cancel{d\lambda = 0} = 0$$

$$= \int d\varphi \cdot F(f_t(x))\, \rho\, d(\text{Leb}) \;\ominus$$

$\Leftarrow\!\!>$ $\int \frac{\partial\varphi}{\partial x_k}\left(F_k(f_t(x))\cdot\rho\right) d\,\text{Leb} = 0$ integration by part

$$\int \frac{\partial}{\partial x_k}(\rho F_k)\cdot\varphi \; d\,\text{Leb} = 0$$

$\Leftarrow\!\!>$ $\frac{\partial}{\partial x_k}(\rho F_k) = 0$

$(\rho = 1)$

<u>Cor</u>. local flow preserve volume on an open set $\subset\mathbb{R}^n$

iff. $\frac{\partial}{\partial x_k} F_k = 0.$ divergence $=0$.

<u>Cor</u> Hamilton system preserves Leb measure

$$\Omega \subset \mathbb{R}^{2n} = \{ q_1, \cdots, q_n, p_1, \cdots, p_n \}$$

i-th atom

position speed

$H: \Omega \to \mathbb{R}$ C^∞ map. Hamiltonian

$$\begin{cases} \dfrac{\partial q_i}{dt} = \dfrac{\partial H}{\partial p_i} \\[2ex] \dfrac{\partial p_i}{dt} = -\dfrac{\partial H}{\partial q_i}. \end{cases} \qquad F_k = \begin{cases} \dfrac{\partial H}{\partial p_k}, & k \le n \\[2ex] -\dfrac{\partial H}{\partial q_{k-n}}, & k > n \end{cases}$$

$$\sum_{1\le q\le 2n} \partial_k F_k = \sum_{1\le k\le n}\left(\frac{\partial}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_k}\frac{\partial H}{\partial q_k}\right) = 0$$