Exercise sheet 7

Exercise 1 *(1) Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $f : X \to X$ be a measurable map that preserves μ. Assume f is bijective and its inverse is measurable. Let $\phi : X \to \mathbb{C}$ be a μ-summable function. For every x in X, we define*

$$S_n\phi(x) = \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x)) \quad \text{and} \quad S_{n,-}\phi(x) = \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^{-k}(x)).$$

Show that for μ-almost every x in X, the limits

$$\tilde{\phi}(x) = \lim_{n\to\infty} S_n\phi(x) \quad \text{and} \quad \tilde{\phi}_-(x) = \lim_{n\to\infty} S_{n,-}\phi(x)$$

exist and are equal.

(2) We now take $X = \mathbb{T}^N$, $\mathcal{B}$ the Borel σ-algebra, and $\mu = d\theta$ the Haar measure. Let $E = \mathbb{R}^N$, $p : E \to X$ the canonical projection, $\|\cdot\|$ a norm on E, and d the quotient distance on X, given by $d(\theta, \theta') = \inf\{\|v - v'\| \mid p(v) = \theta, p(v') = \theta'\}$. Let $\psi : X \to \mathbb{C}$ be a Lipschitz function with Lipschitz constant $Lip(\psi)$.

Show that for every θ in X and v in E, we have

$$|\psi(\theta + p(v)) - \psi(\theta)| \leq Lip(\psi)\|v\|.$$

(3) Let M be an $N \times N$ matrix with integer coefficients such that $\det M = \pm 1$, *and let f_M be the transformation of X such that $p \circ M = f_M \circ p$.*

1. *Show that f_M is bijective. We henceforth take $f = f_M$.*
2. *For θ in X, let $\tau_\theta : X \to X$ be the translation by θ. Show that*

$$f \circ \tau_\theta = \tau_{f(\theta)} \circ f.$$

(4) We now assume that M is hyperbolic. We choose for $\|\cdot\|$ a norm adapted to M, and we denote $E = E^s \oplus E^u$ the decomposition of E into M-invariant subspaces such that

$$ch(M) = \max\{\|M|_{E^s}\|, \|M^{-1}|_{E^u}\|\} < 1.$$

1. *Show that for every θ in X and v in E^s, we have*

$$|S_n\psi(\theta + p(v)) - S_n\psi(\theta)| \leq \frac{Lip(\psi)}{n(1 - ch(M))}\|v\|.$$

2. *Show that for every v in E^s, we have $\tilde{\psi} \circ \tau_{p(v)} = \tilde{\psi}$ μ-almost everywhere.*

3. *Show that for every v in E^u, we have $\tilde{\psi} \circ \tau_{p(v)} = \tilde{\psi}$ μ-almost everywhere.*
4. *Show that for every v in E, we have $\tilde{\psi} \circ \tau_{p(v)} = \tilde{\psi}$ μ-almost everywhere.*
5. *Show that $\tilde{\psi}$ is constant μ-almost everywhere.*
6. *Show that $\tilde{\phi}$ is constant μ-almost everywhere.*
7. *Deduce that f is ergodic for μ.*