Exercise sheet 5

Exercise 1 *Let $p \in]0,1[$, $q = 1-p$, $X = \mathbb{R}/\mathbb{Z}$ and let $f : X \to X$ be the doubling map $x \mapsto 2x$ on the circle. We identify the set X with the set $[0,1[$. For every Borel probability measure μ on X, we denote $\mu_1 = T(\mu)$ as the Borel probability measure on X defined by*

$$T(\mu)(A) = q\,\mu(f(A \cap [0, \tfrac{1}{2}[)) + p\,\mu(f(A \cap [\tfrac{1}{2}, 1[))$$

for every Borel set A of X. We denote $\mu_n = T^n(\mu)$.

For all integers k, d such that $0 \leq k < 2^d$, we denote by $I_{k,d}$ the interval $I_{k,d} = [\frac{k}{2^d}, \frac{k+1}{2^d}[$, we write k in base 2:

$$k = \sum_{i=0}^{d-1} a_i 2^i$$

with $a_i = 0$ or $a_i = 1$, and we denote $w_k = \sum_{i=0}^{d-1} a_i$.

- *1) Compute $\mu_1([0, \frac{1}{2}[)$ and $\mu_1([\frac{1}{2}, 1[)$.*

  2) Show that we have $\mu_d(I_{k,d}) = p^{w_k} q^{d-w_k}$.

  3) Show that, for $n \geq d$, we have $\mu_n(I_{k,d}) = p^{w_k} q^{d-w_k}$.

  4) Show that if a subsequence $(\mu_{n_i})_{i \in \mathbb{N}}$ converges weakly to a Borel probability measure μ_∞, then, for all $k < 2^d$, we have $\mu_\infty(I_{k,d}) = p^{w_k} q^{d-w_k}$. (One may first show that $\mu_\infty(\{\frac{k}{2^d}\}) = 0$.)

  5) Show that there exists a unique Borel probability measure $\nu = \nu_p$ on X such that, for all k, d with $k < 2^d$, we have $\nu(I_{k,d}) = p^{w_k} q^{d-w_k}$.

  6) Show that, for every Borel probability measure μ, the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges to this probability measure ν.

  7) Determine ν when $p = \frac{1}{2}$.

- *1) Show that ν is the unique Borel probability measure on X such that $T(\nu) = \nu$.*

  2) Show that ν is f-invariant.

  3) Show that, for every Borel set A of X, we have $\nu(f^{-d}(A) \cap I_{k,d}) = \nu(A)\nu(I_{k,d})$.

  4) Show that f is mixing for ν.

  5) For every x in X (identified with $[0,1[$), we denote $x = 0.b_1 b_2 b_3 \ldots$ the dyadic expansion of x, in other words,

$$x = \sum_{i \in \mathbb{N} \setminus \{0\}} b_i 2^{-i}$$

with $b_i = 0$ *or* $b_i = 1$*. Compute, for* ν*-almost every* x *in* X*, the limit*

$$\lim_{n\to\infty} \frac{1}{n} Card\{i \leq n / b_i = 1\}.$$

6) Show that, for ν*-almost every* x*, the orbit* $\{f^n(x)/n \in \mathbb{N}\}$ *is dense in* X*.*

7) Show that if $p \neq p'$*, then the probability measures* ν_p *and* $\nu_{p'}$ *are mutually singular (i.e., there exists a set that has measure zero for one and full measure for the other).*