Exercise sheet 4

Exercise 1 *Let M be a $N \times N$ matrix with integer coefficients such that* $\det M \neq 0$. *We suppose that $|\lambda_i| \neq 1$ for all i, where $\lambda_1, \cdots, \lambda_N$ are eigenvalues of M. Denote $\mathcal{N}(n, M)$ the number of fixed points of M^n on $\mathbb{T}^N$, that is $M^n x = x$.*

- *Show that*
$$\mathcal{N}(n, M) = \prod_{j=1}^{N} |\lambda_j^n - 1|.$$

- *Compute the value of* $\lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}(n, M)$.

Exercise 2 *Let $f : [0, 1] \to [0, 1]$ be the baker's map defined by $f(x) = 2x$ if $x \leq 1/2$ and $f(x) = 2(1 - x)$ if $x \geq 1/2$. Show that f preserves the Lebesgue measure λ and λ is mixing for f.*

Exercise 3 *Let X be a compact metric space, f a continuous map and μ a f-invariant Borel probability measure on X with support equal to X. If μ is mixing for f. Show that f is topogical mixng, that is for any two open sets U, V in X, there exists n_0 such that for all $n \geq n_0$, we have $f^n(U) \cap V \neq \emptyset$.*

Exercise 4 *Let $(X, \mathcal{B}, \mu)$ be probability space and f a measurable map preserving μ. Suppose f is mixing for μ.*

- *Show that the map $f \times f : X \times X \to X \times X$ is mixng for the product measure $\mu \times \mu$.*

- *Show that f is weakly mixing.*

Exercise 5 *Show that a circle rotation R_α on $\mathbb{R}/\mathbb{Z}$ is not mixing. Show that $R_\alpha \times R_\alpha$ is not ergodic.*

Exercise 6 *Let $(X, \mathcal{A}, \mu)$ be a probability space and $f : X \to X$ a measurable measure-preserving map. For $k \in \mathbb{N} \setminus \{0, 1\}$, we say that f is* mixing of order k *if for all measurable subsets $A_1, \ldots, A_k$ of X, and all sequences $\tau_1, \ldots, \tau_k$ from $\mathbb{N}$ to $\mathbb{N}$ such that $\lim_{n\to\infty} \tau_{i+1}(n) - \tau_i(n) = +\infty$ for $i = 1, \ldots, k - 1$, we have*

$$\mu\left(\bigcap_{i=1}^{k} f^{-\tau_i(n)}(A_i)\right) \xrightarrow[n\to\infty]{} \prod_{i=1}^{k} \mu(A_i).$$

1. *Show that a symbolic dynamical system is mixing of order k for all $k \in \mathbb{N} \setminus \{0, 1\}$.*

2. *Let $N \in \mathbb{N} \setminus \{0, 1\}$. Show that the map $f_N : \theta \mapsto N\theta$ on the circle $\mathbb{R}/\mathbb{Z}$ is mixing of order k, for all $k \in \mathbb{N} \setminus \{0, 1\}$, with respect to the Haar measure.*

3. *Let* $N \in \mathbb{N} \setminus \{0\}$ *and let* M *be an invertible* $N \times N$ *matrix with integer entries, having no eigenvalue which is a root of unity. Show that the map* $f_M : \mathbf{x} \bmod \mathbb{Z}^N \mapsto M\mathbf{x} \bmod \mathbb{Z}^N$ *on the torus* $\mathbb{R}^N/\mathbb{Z}^N$ *is mixing of order* k*, for all* $k \in \mathbb{N} \setminus \{0, 1\}$*, with respect to the Haar measure.*