Exercise sheet 3

Exercise 1 *Let f be a unique ergodic transformation on a compact metric space X and μ is the unique f-invariant measure on X with support μ.*

- *Prove that any f-invariant non empty closed set of X contains S.*
- *If f is bijective, do we always have the equality $X = S$?*

Exercise 2 *Show the map $f(x) = (x_1 + \alpha, x_1 + x_2, \cdots, x_{n-1} + x_n)$ on $\mathbb{T}^n$ preserves the Lebesgue measure.*

Exercise 3 *On the torus $\mathbb{T}^2$, let $f(x, y) = (x + \alpha, y + \psi(x))$ and $f_0(x, y) = (x + \alpha, y)$ with ψ continuous. Let $h(x, y) = (x, u(x) + y)$. Suppose u satisfies the equation*

$$\psi(x) = u(x + \alpha) - u(x).$$

Show that if u is continuous (measurable), then h is a continuous (measurable) conjugate between f and f_0.

Exercise 4 *Let H be a Hilbert space and $U : H \to H$ a unitary transformation. Let H^U be the closed subspace $\{h \in H \ |U(h) = h\}$ and p^U the orthogonal projection on H^U. We want to prove that*

$$\forall h \in H, \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k(h) = p^U(h). \tag{1}$$

- *Show (1) for h in H^U.*
- *Show (1) for $h = U(h_0) - h_0$ for h_0 in H.*
- *Show the subspace $(U - Id)H$ is dense in the orthogonal of H^U.*
- *Conclude.*