Exercise Sheet 2

Exercise 1 *Let $(X, \mathcal{B}, \mu)$ a measurable space and f a measurable map on X which preserves the measure zero sets (i.e. For all $A \in \mathcal{B}$, if $\mu(A) = 0$ then $\mu(f^{-1}A) = 0$).*

Prove that if a measurable map $\varphi : X \to \mathcal{C}$ is almost everywhere f-invariant then there exists a measurable map φ' such that φ' is f-invariant and equal to φ almost everywhere.

Exercise 2 *Let $(X, \mathcal{B}, \mu)$ a probability space and f a measurable map preserving μ. Prove that f is ergodic if and only if for all A, B in $\mathcal{B}$ with $\mu(A), \mu(B) > 0$, then there exists $n \geq 1$ such that $\mu(f^{-n}(A) \cap B) > 0$.*

Exercise 3 *Let $(X, \mathcal{B}, \mu)$ a probability space. Show that the shift map σ on X^E is ergodic for the measure $\mu^{\otimes E}$ with $E = \mathbb{N}$ or $\mathbb{Z}$.*

Exercise 4 *Let $(X, \mathcal{B}, \mu)$ a probability space and f a measurable map preserving μ and ergodic. Prove that for all $A, B \in \mathcal{B}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(f^{-k}(A) \cap B) = \mu(A)\mu(B).$$

Exercise 5 (Kac recurrence theorem) *Let $(X, \mathcal{B}, \mu)$ a probability space and f a measurable map preserving μ and ergodic. Let A be a measurable set with $\mu(A) > 0$. For $x \in X$, let*

$$n_A(x) = \inf\{n \geq 1,\ f^n(x) \in A\}$$

be the first return time to A. Show that

$$\int_A n_A(x) d\mu = 1.$$