Autumn 2025 Exam: Dynamical systems

You can use all the results from the course and please write clearly the name or the content of the result. Each sub-question has 10 points.

Exercise 1 (Unique ergodicity)

- *Give a definition of unique ergodicity of a continuous dynamical system $f : X \to X$ on a compact metric space.*
- *Give one example of uniquely ergodic system.*
- *If (f, X) is a uniquely ergodic system, show that for any continuous function φ, the $S_n\varphi(x)$ converges uniformly for $x \in X$.*

Exercise 2 (Symbolic dynamics) *Let $\Lambda = \{1, 2, \cdots, k\}$ be a finite set, $\Omega = \Lambda^{\times\mathbb{N}}$ and S is the shift map on Ω given by $S(x_n)_n = (x_{n+1})_n$.*

- *What are the periodic points of S?*
- *Let P_n be the set of periodic points of period n (that is $S^n x = x$), what is $\#P_n$, the cardinality of P_n?*
- *What is the weak limit of the following measures?*

$$\lim_{n\to\infty} \frac{1}{\#P_n} \sum_{x\in P_n} \delta_x$$

- *If we let P'_n be the set of primitive periodic points with period n (that is if $S^m x = x$ then n divides m), show the same convergence as the previous one when replacing P_n by P'_n.*

Exercise 3 (Entropy) *We write T_p for the times p map on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let μ be a T_2 invariant measure on $\mathbb{T}$.*

- *Show*

$$h_{(T_3)_*\mu}(T_2) \leq h_\mu(T_2)$$

- *Show*

$$h_\mu(T_2) = h_{(T_3)_*\mu}(T_2)$$

Exercise 4 (Torus and Subtorus) *Let $M = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, consider the map f_M on the torus $\mathbb{T}^3$ given by*

$$f_M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mod \mathbb{Z}^3.$$

- *Is the Lebesgue measure on $\mathbb{T}^3$ ergodic for f_M?*
- *For any $t \in \mathbb{T}$, show the measure $d\mu_t = dxdy$ on the subtorus $z = t$ is ergodic for f_M.*
- *Show it is also mixing for f_M.*

What about the same questions for $M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$*?*